Spatial Skills and Mathematical Problem Solving Ability in High School Students

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Project Rationale

My master's project question is “how do spatial skills affect a student's problem solving ability in a high school mathematics class?” I am passionate about this question because it is something I have wondered about since I began my study of university-level mathematics. Namely, I quickly noticed a commonality between all of the most gifted mathematicians I met during my studies: they all had uncanny geometric intuition. When they solved problems, they would often wait several minutes before writing anything down on paper. But then when they began recording their thoughts on paper, it would often just be a quick geometric sketch accompanied by either a geometric argument or an argument inspired by a geometric idea. Further, I also noticed that the trend in modern mathematics is to reframe most difficult questions in geometric terms. While there may be many explanations for this trend, anecdotal evidence points to this being done because people can best understand abstract mathematical concepts on a proto-linguistic, geometric level. I focused my research question on spatial skills rather than the hard-to-define term “geometric intuition” because it seemed to me as if the important part was not geometric shapes but geometric relationships—that is, spatial relationships. As I've moved away from academia and closer to secondary education, my interest in this question has not waned.

Indeed, my experience in the classroom has pointed to the potential importance of spatial skills in the mathematics classroom. Namely, there seems to be a benefit for students who learn mathematical methods that are meant to stress geometry over rote memorization. For example, I once spoke to a group of students that discussed how they learned the “area model” of
multiplication as a means to replace long multiplication and the FOIL method.\textsuperscript{1} Of course, even
geometric models are susceptible to being reduced to sheer memorization—some of these students
said that they did not know why the area method worked or why it was any better than FOIL or the
traditional “long multiplication” algorithm: for them, the methods were equivalent because they
simply required the memorization of a few rules. This led me to believe that these students were
taught the process of “area method” for multiplying numbers or polynomials, but were never
taught the spatial interpretation underlying it. Alternatively, they may have learned both the
process and underlying spatial interpretation of the method, but only remembered the former part
of it since this was all they needed to begin solving problems. This led me to the hypothesis that
students who actually use spatial skills when solving math problems would likely be more
successful when solving math problems. After finalizing the focus of this project, I realized that if
my results showed a positive correlation between spatial skills and problem solving ability, it
would be reasonable to propose further research on whether teachers can help to develop content-
specific spatial skills in students. For example, the students might complete daily spatial tasks that
could help students maintain their geometric understanding of fundamental mathematical problems
so that they do not reduce the subject to a set of arbitrary rules.

By finding evidence of there being a positive correlation between a student's spatial skills
and problem solving ability, I can show that teachers may need to make spatially-oriented problem
solving techniques more prevalent in the classroom. Developmental psychologists have shown that
students begin to develop the ability to comprehend abstract mathematical concepts and perform

\textsuperscript{1} The “FOIL” method of multiplication is an acronym for “First, Outside, Inside, Last” that is meant to help
students remember how to correctly multiply two binomial expressions.
advanced spatial manipulations during early adolescence. However, according to the theories of Piaget, while average adolescents become developmentally ready for these abstract tasks around the age of 14, they will never reach their full potential (the formal operations stage) if they don't cultivate those abilities (1999, pg. 153). So if students can increase their mathematical performance by developing their spatial abilities in a mathematical context, then it might even be the case that taking a spatial focus in a math class might help more students reach this formal operations stage. It might also be the case that spatial skills are innate and that these skills cannot be taught or developed to any great extent. If this is the case, then the results of my project will still be valuable because this implies that teachers might need to differentiate their instruction so that course material is equally accessible to both spatial and non-spatial problem solvers.

However, since this question is one that I have a personal interest in helping to resolve, it is not crucial whether my results can be used to make a definitive conclusion or to recommend universal policy recommendations in mathematics education. Rather, I feel that my research will help me improve my own practice throughout my teaching career; my results will apply to my own classroom, my philosophy of education, and my knowledge of the subject. Also, I view this project as a part of an ongoing investigation of how students best learn mathematics and how I as a teacher can help facilitate that learning. This initial investigation in the form of a master's project will help me to learn whether there exists a correlation between a student's spatial skills and math ability, but the ongoing investigation will help me learn whether it would be valuable to concentrate on teaching students spatial problem solving skills, such as how to draw a good geometric diagram of an algebraic situation, how to think about abstract algebraic concepts from a geometric perspective,
and how to use one's spatial skills to start solving a math problem before even writing anything down on paper.

Further, this might affect how I approach curriculum development later in my career. If spatial reasoning truly is as important to problem solving success in a mathematics classroom, then I might experiment with teaching an “integrated” curriculum—that is, instead of a sequential curriculum that covers algebra in freshman year, geometry in sophomore year, trigonometry in junior year, and calculus in senior year, it might be more logical to interweave all of these subjects together so that geometry and geometric ideas can be made a greater part of all of these subjects.
Research Question Development

My master's project question has gone through quite a bit of development since its initial inception. Unlike other researchers, my question did not undergo a slow evolution. Rather, it underwent a drastic and rapid change that, while the original integrity of my first question was retained, the way in which I conducted my research completely shifted. At first, my question was “Will students perform better in a mathematics classroom and feel more confident with the subject if they spend five minutes every day performing mental exercises such as visualizing geometric objects, mentally interpreting a graph, or performing mental arithmetic?” The goal of this question was to discover whether having students practice spatial activities would improve their math ability. Originally, I envisioned my project being structured as follows:

First, I hoped to do my research in the classroom of a teacher who had two sections of the same class. Ideally, I wrote that I would like to perform this study at a non-magnet school with standard 50 minute class periods. I originally said that precalculus or algebra/trigonometry students would be a good match for this study—the material they would learn would have complemented the kinds of exercises I wanted them to perform, and most of the students would be at least 15 years old (the age at with Piaget says that they will be best prepared for abstract mathematical work). Next, I wanted to try as hard as possible to create some sort of control group in my study. Since there are more variables in a high school classroom than can be controlled, I called my control group a “comparison group.” Namely, I planned to do exercises with one group but not the other so that I have a basis for comparison and drawing conclusions from my data.

My preliminary procedure for data collection began with visiting my experimental group in
class every day and go through five minutes of mental exercises with the students for a period of at least 5 weeks. I had a sample of some exercises that may be repeated throughout this period, such as the following:

Cube comparison test: Participants are asked whether two cubes are identical after an appropriate series of rotations or whether they are distinct.

Paper folding test: Participants watch as the researcher folds a sheet of paper in three places and punches holes into the paper. The participants have to identify how the paper will look when unfolded.

Hidden patterns test: Participants have to “determine if a given figure is embedded in a geometric pattern.” (Velez, Silver, and Tremaine 2006, pg. 2)

The rules of these exercises were meant to be somewhat intuitive so that the students could concentrate on actual mental manipulations rather than the semantics of the task. Further, they were designed to start out easy and get progressively harder as the weeks went by. For example, during the first week, students may have been asked to mentally tabulate how many cubes there are on a picture of 5 cubes that was handed out to each of the students, while they might have been required to tabulate several dozen cubes towards the end of the experiment.

By pre-testing and post-testing in both classes on the particular skills I wanted to identify, I thought I could try to determine whether the exercises resulted in measurable growth in the experimental group. There are several tests already available for assessing spatial ability, so I did not think that I would need to write one myself. If the final results showed inconsistencies within the experimental group, I thought I could try to determine whether there was an unexpected variable that I could identify, such as the students' preferred learning styles. Then I planned to look at student work and assessments (particularly grades) in both classes to see if the exercises could have been linked to a general increase in performance in the experimental group.
While this research still intrigues me, my master's project question evolved as I discovered how difficult it would be to complete this study. First, even though this version of the project only required working with a group of students for five minutes every day, this is still more time than most teachers are willing to give up—for most teachers, five minutes a day would be enough to take up more than ten percent of their total class time with their students. Further, the gains that I was expecting from just five minutes of spatial activities a day would probably not be quantifiable until many more than five weeks of practice. This is particularly true because I wanted to have the students complete spatial tasks that were unrelated to the course material they were studying in the class. That is, the spatial activities would not give them any problem solving skills or content knowledge that could help them in their math classes: it really would be nothing more than some mental exercise before officially starting math class.

As a result, I modified my question by making it “How do spatial skills affect a student's problem solving ability in a high school mathematics class?” This question still made it possible to explore my interest in the relationship between spatial skills and math ability. However, this question allowed me to collect data in a more realistic manner. In addition, it was more open ended so that no matter what my data said, I could still write a meaningful master's project. With my original question, I was set up for disappointment if there did not end up being any noticeable correlation between performance in a math class and working on daily spatial exercises. In addition, by changing my question, I gained flexibility in how I chose to collect data. Instead of a rigorous regiment of pre-testing math ability, post-testing math ability, and repeating the same process for spatial ability, I opened my question up to administering student surveys and
conducting focus group interviews. These methods for collecting data have allowed me to draw a colorful, detailed picture of how the students I worked with went about solving problems and using spatial reasoning.
Literature Review

Part 1: Introduction

For many years, anecdotal evidence from mathematicians has suggested that there is a direct link between a student's spatial skills and his or her mathematical ability. Stories have circulated about mathematicians who solve entire problems in their minds with the absence of an inner dialog; instead, they “see the solution in their minds” in the form of moving shapes, colors, or textures. Popular media outlets publish newspaper articles and air television specials about savants who can perform extraordinary mathematical feats such as computing cube roots faster than a calculator; they experience this computation as a mental flash of colors and shapes (Johnson, February 12, 2005). As a result, the notion of there being a link between spatial skills and mathematical ability has entered the annals of “common knowledge.” While this anecdotal record supports the link between superb spatial skills and extraordinary ability in mathematics, researchers have also been fascinated by the possibility of there being a direct connection between these two abilities in average people, especially adolescents in a math class.

Despite the fact that the link between raw spatial ability and mathematical performance has been studied extensively, conclusive results from this research have been somewhat elusive. For example, Casey, Nuttal and Pezaris (1992) found that, in their sample, spatial ability was only linked to mathematical achievement in males and left-handed females. This is a weak correlation at best. In contrast, Ferrini-Mundy (1987) found a negative correlation between spatial ability and math performance; she discovered that the men she studied were more likely to have strong spatial visualization abilities, while the women in the study were more likely to be successful in
mathematics. Namely, the students who were most likely to have strong spatial abilities were also the most likely to be the least adept at calculus.

Fennema and Tarte (1985) note in an article that the inconsistency in previous studies has stemmed from the inherent difficulty in conducting a correlational study about spatial and mathematical ability. This is largely due to the fact while one might be able to establish a “...direct relationship between spatial visualization and mathematical tasks that are overtly spatial,” such a result does not necessarily generalize “...to a broader spectrum of mathematics” (p. 184). From this perspective, one might try to find a more effective method of studying the connection between spatial skills and general mathematical ability. Or, alternatively, one can embrace this original methodology, limitations and all, when studying what it says about teaching, learning, and problem solving.

In this way, my focus in this literature review will be on three areas: (a) problem-solving skills in a mathematics classroom, (b) how spatial skills affect the problem-solving process, and (c) how educators can use visualizations and manipulatives to allow all students access to spatially-oriented mathematical tasks. I will discuss these domains in this order and in relation to my master's project question, which asks about how spatial skills affect a student's problem solving ability in a high school mathematics class. I feel it is important to note the final domain of my literature review about using manipulatives and digital visualizations in order to help students complete mathematical activities that have a spatial components is crucial; if students are not given access to these types of mathematical problems while they are students, they may never master these skills. Indeed, according to the theories of Piaget (1999), while average adolescents become
developmentally ready for abstract (spatially-oriented) tasks around the age of 14, they will never reach their full potential (the formal operations stage) if they do not cultivate their abilities (p. 153). Although some students may be able to cultivate their spatial abilities using their innate talents, others may need to utilize manipulatives or visualization software to develop their spatial skills.

**Part 2: The Art of Problem Solving**

While the study of problem solving might seem to be a rather straightforward pursuit, the academic research on the subject is quite factional. Cognitive scientists, educational researchers, and psychologists have published numerous studies and position papers on what problem solving is and whether it should be taught in schools, many of which conflict with one another. Perhaps the first issue is to define problem solving as the process one goes through when one pursues a task-based or goal-based activity. While people encounter problem solving tasks in countless different situations, problem solving has a particular relevance in a mathematics classroom. First of all, problem solving is viewed as a skill that students must develop in order to succeed in an academic setting. But within an academic setting, problem solving becomes even more relevant in a mathematics classroom because, by its very nature, the study of this subject focuses on solving clearly defined problems in an axiomatic system. While problem solving skills might be an implicit benefit in other subjects, mathematics overtly demands the presence of these abilities. Whether in a group setting or an individual testing situation, students must exhibit an ability to solve problems in order to succeed in any math class.

In Boekaerts, Seegers and Vermeer (1995), problem solving is described as a context-
dependent activity, in which “[e]nvironmental variables interact with person variables to shape students' behavior as they work on mathematical tasks” (p. 241). That is to say, the process of problem solving is not self-contained and, like many aspects of classroom learning, is affected by countless variables related to a student's environment, past experience, and personal attributes. This becomes even more complex when problem solving becomes “a social activity in the context of the classroom” (1995, p. 241). These authors continue on to distinguish between two different aspects of problem solving. They define “cognitive aspects” of problem solving to be the knowledge-based skills related to remembering mathematical definitions, theorems, algorithms, and useful procedures. Also included in the category of the cognitive aspects of problem solving are “metacognitive skills” about how one goes about using or planning to use knowledge-based skills. In contrast, “affective aspects” of problem solving relate to the “influence of emotions during problem solving.” Positive affective factors include when a student has a sense of confidence related to his or her problem solving ability, has expectations of success, or has a desire to “persist in the face of difficulties” (1995, p. 242). On the other hand, affective factors can negatively influence a student's problem solving experience if he or she expects to fail, feels angry when solving problems, or becomes easily frustrated. Boekaerts, Seegers, and Vermeer note that negative affective factors can lead to “interruptions of planned sequences of thought or action,” which they define to be a “blockage” in the problem solving process (1995, p. 243).

By concentrating on the affective aspects of problem solving, these researchers decided to adopt a “model of adaptable learning,” which says that a student's affective problem solving skills are greatly influenced by his or her goals in the classroom. Students who view classroom success
as crucial to success and well-being are more prone to a negative affective response, while students who view learning as a chance to increase “competence for reasonable costs” are more likely to have a positive affective response.

In contrast, many researchers have focuses on cognitive skills necessary to effective problem solving. However, there is much debate even within the domain of the cognitive problem solving skills. Sweller claims that while most researchers thought it was important to teach a general approach to effective problem solving, it is actually more effective to concentrate on “domain-specific knowledge,” because it is this knowledge that “differentiates novices from experts” (Sweller, 1990, p. 412). In essence, Sweller is advocating for a focus on cognitive rather than metacognitive skills when teaching students to be effective problem solvers; while cognitive skills involve specific knowledge related to the subject, metacognitive skills are involved with the more general strategies of monitoring one's own thought process during the problem solving experience. He supports this claim by citing a long-term study published in 1988 by Swing, Stoiber, and Peterson which showed that high-ability students showed no improvement after being taught general problem-solving strategies, while these strategies actually hurt performance in low-ability students (p. 413). Lawson (1990) refutes this claim and says that when combined with a “well-organized knowledge base,” teaching metacognitive problem-solving strategies can help improve transfer and performance (p. 409). Examples of such strategies include “goal setting, monitoring, allocation of attention, and selection of more specific processing operations,” which Lawson calls “executive strategies.” Lawson also refers to “domain-specific strategies,” which are slightly more general than knowledge-based cognitive skills described in Boekaerts, Seegers, and
Vermeer; in the domain of a math class, the strategy might be to try drawing a diagram of a mathematical situation or to create a table to help keep track of important data in a problem. Finally, Lawson defines “task orientation strategies” in much the same way that affective factors were described earlier: the part of problem solving that is influenced by a student's attitude or expectations (1990, p. 404).

Hart (1993) outlines how these different aspects of problem solving can contribute to an either positive or negative experience. Hart points out that, from a cognitive standpoint, the problem solving experience usually goes awry when students have a difficult time relating the task to relevant past experiences; she calls this a “lack of experiential framework” (p. 169). This supports Sweller's notion that it is domain-specific knowledge and experience with many different kinds of problems that helps one complete mathematical tasks successfully. When students lack this knowledge and experience, this can lead to students imposing “unrequired restrictions” on the problem that make the problem more difficult to solve. An colloquial example of this phenomenon is observed in a popular riddle: “A man and his son are in a horrific car accident. The man dies instantly, but his son survives and is rushed to an area hospital. Upon looking at the boy, the doctor says, 'I cannot operate on him, he is my son.' How can this be?” The answer, of course, is that the doctor is the boy's mother. However, many respondents put the unrequired restriction on the riddle that the doctor must be a male, most likely due to their experiential framework and subsequent assumptions about the world. Hart also notes that metacognitive roadblocks also exist in the form of weak self-regulation and a lack of reflective thought. Namely, Hart claims that, ideally, students should consciously reflect on their problem solving: “Why am I using this technique” or “How
will this strategy help me solve this problem?” Finally, she also outlines that affective factors can hinder problem solving ability. Some examples of this might be developing a negative attitude towards the task, their abilities, or even just being confused about what the task required them to do (1993, p. 169). On the other hand, Hart describes collaboration and monitoring activities that occur in the context of a small group as factors for improving a student's performance on a problem-solving task (1993, p. 170).

These different perspectives on the process of problem solving have particular relevance to my own research project. First of all, instead of simply studying the relationship between spatial skills and problem solving ability, it seems important to recognize that spatial skills have an effect on both cognitive and affective aspects of problem solving. From Lawson's point of view, the integration of spatial skills into a student's repertoire of problem solving strategies could be classified as a domain-specific strategy, particularly when students use these skills to draw a diagram or create a graph meant to help describe the problem on which they are working. However, there are affective factors involved as well: students who have an aversion to spatial reasoning might give up easily on a problem that requires these skills, or a student who tries to use spatial reasoning as much as possible might become frustrated when a problem is too complicated for a spatial model or has no obvious spatial interpretation. From this perspective, the study of spatial skills and problem solving has two major components: the raw spatial ability of the student working on a mathematical task, and the ability of students to use/apply those skills during that task by using affective and cognitive problem solving skills.
Part 3: Applying Spatial Skills to Problem-Solving Tasks

Before discussing the connection between spatial skills and problem-solving ability, I feel that it is crucial that I define key terms related to this discipline. This is necessary because the terminology used by one researcher often has subtle differences from the terminology of another. For example, van Garderen (2006) differentiates between “spatial ability,” “visual imagery,” “mental imagery,” “spatial visualization,” and “spatial imagery.” While these terms seem somewhat synonymous, they are quite distinct in van Garderen’s view. The two definitions that the author concentrates on the most are “visual imagery” and “spatial imagery.” The former refers to “...the representation of an object such as its shape, color or brightness,” while the latter refers to “...the representation of spatial relationships between the parts of an object and the location of objects in space or their movement,” (2006, p. 497). While these definitions are logical, they are markedly more detailed than what I have used in my own research.

Instead of distinguishing between “visual imagery” and “spatial imagery,” the focus of this research is instead be on whether students use imagery, and whether or not that imagery encodes spatial information. More specifically, I have tried to determine whether the imagery used embeds the geometric relationships between the objects in question. In contrast to van Garderen’s study, though, I tried to limit the extent to which I connected the “quality” of the student's imaging techniques to their performance. In short, instead of adopting van Garderen’s view that imagery which encodes extraneous information such as color or texture of geometric objects is inherently negative, I focused on whether students used imagery that encodes spatial relationships, regardless of the imagery's other features.
To further complicate terminology, I have found that McGee (1979) defines “spatial visualization” as the “...ability to mentally rotate, manipulate, and twist two- and three-dimensional stimulus objects” (p. 896). He also defines “spatial orientation ability” as the “...comprehension of the arrangement of elements within a visual stimulus pattern” and “...the aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented” (1979, p. 891).

Further, Silver, Tremaine, and Velez (2006) write about how different kinds of spatial ability can affect how well someone can construct useful visualizations (p. 1). For example, they discuss how MRI data takes a three-dimensional image of the brain and projects it into a two-dimensional version of that original image. However, in order for it to be useful, one has to have spatial skills that are developed enough to reconstruct the original picture from that projection. While this study does not specifically focus on the potential connection between mathematics and spatial ability, they do introduce many measures they used for determining the different kinds of spatial ability a person might exhibit:

- **Cube comparison test:** Participants are asked whether two cubes are identical after an appropriate series of rotations or whether they are distinct.
- **Paper folding test:** Participants watch as the researcher folds a sheet of paper in three places and punches holes into the paper. The participants have to identify how the paper will look when unfolded.
- **Hidden patterns test:** Participants have to “determine if a given figure is embedded in a geometric pattern.” (2006, p. 2)

These tests are meant to break down spatial ability into six major components, including the ability to recall the position of an object in an array, the ability to predict the path of a projectile, the ability to visualize an object being reflected and rotated in space, and the ability to identify a
subobject of a more complicated object.

In contrast to the numerous and subtle delineations in these works, I have chosen not to differentiate between different subsets of spatial skills. This is in an effort to maintain order within my own research and so that I could remain focused on the spatial core of my research question, rather than on subtleties and semantics. In short, I have instead chosen to concentrate whether students choose to supplement their problem-solving technique with the use of spatial imagery, rather than study its exact subcomponents.

It is also important to note that while researchers still debate the role of spatial skills in general mathematical ability, they have found a large amount of evidence to support the fact that spatial visualization skills play an important role in solving certain kinds of mathematical problems (Fennema and Tartre, 1985, p. 184). In many ways, this is a logical notion since many math problems are written in order to guarantee that they have spatial significance. For example, problems from calculus and geometry often explore the relationship between space and time. Early in a student’s geometry career, he or she learns how to compute the area of a shape, the surface area of a surface, and the volume of a solid. As that student progresses to a calculus classroom, he or she will explore the notion of adding another dimension to these problems by studying the change in area of a shape or the volume of a solid over time. In addition, spatial skills help students solve problems that require “...the use of diagrams and charts” (Booth and Thomas, 2000, p. 169).

Even studies that did not find a correlation between problem-solving ability and spatial ability noted an important distinction between how spatial versus non-spatial thinkers solved math problems: Brown and Wheatley found that while fifth-grade girls were just as likely to solve a
problem correctly whether they solved it using spatial skills or not, those girls who utilized spatial skills had a “relational” understanding of the subject. Namely, their understanding of mathematics came from knowing the relationships between various algebraic and geometric structures. On the other hand, fifth-grade girls who lacked strong spatial skills were more likely to have an “instrumental” understanding of the subject—they simply memorized the course material, and exhibited little conceptual knowledge of the subject (1989, p. 144). For example, these fifth-grade girls could all complete simple multiplication problems, but only the more spatially-inclined girls could explain the geometric significance of multiplication as a means of computing the area of a rectangle (1989, p. 144). Similarly, Fennema and Tartre found that a student’s spatial ability did not affect his or her likelihood of solving certain math problems correctly, but the students with high spatial ability were still more likely to be “...able to convert word problems to accurate diagrams, and to use those diagrams to get correct solutions” (1985, p. 193).

As previously mentioned, researchers have also noted that the common thread between all effective problem solvers who used imagery was that their thoughts encoded important spatial information about the problem. For example, a student completing a math problem about building a fence on the land around a house would think more about the distances between the fence and the house and the perimeter of the fence rather than the color of the brick on the house or whether it was a picket or chain-link fence. Van Garderen calls the former “schematic imagery,” and claims that it is “more sophisticated” (2006, p. 497). Similarly, Hegarty and Kozhevnikov (1999) claim that schematic visualizers are more successful when solving math problems than students who use visual imagery.
Indeed, Arcavi (2003) writes that the ability to visualize and use spatial reasoning has begun to be viewed as a “key component of reasoning,” problem solving, and even constructing accurate proofs of mathematical phenomena (p. 235). From this point of view, it does not suffice to study how spatial reasoning skills affect the problem solving process of students studying mathematics. Rather, its increasingly central role in the subject implies that educators may need to find methods for making spatial interpretations accessible to all students, not just the ones who demonstrate a penchant for this type of reasoning. In fact, Arcavi suggests that students who lack a grasp on visual/spatial interpretations of mathematical problems and how they relate to “analytic representations of the same situation” might miss out on attaining a “core understanding of mathematics” (2003, p. 235).

**Part 4: Exercises, Visualizations, and Manipulatives**

Keller and Hart (2002) concentrated on how teachers can develop spatial skills in their students, rather than trying to establish a link between those skills and overall mathematical performance. Namely, they used technology to try to help students develop the skills necessary to interpret two-dimensional projections of three-dimensional objects (p. 15). This article demonstrates that, contrary to what many educators might think, adolescents must have already developed their spatial skills to a relatively high level if they can accurately interpret a two-dimensional picture as a representation of a three-dimensional object. This has ramifications for research about the connection between spatial skills and mathematical ability because it demonstrates that teachers might need to treat spatial skills as something that need to be developed, rather than innate skills that cannot be taught to students. Further, this fact shows that educators
may need to use manipulatives in class if they want all students to be able to complete math problems that require spatial skills, since not all student will have the same developmental readiness for abstract spatial representations. For example, not all student may be able to complete a math problem about a cube, not because they cannot understand the concept of a three-dimensional solid, but because some students might not be able to work effectively with a two-dimensional projection of a three-dimensional shape. However, educators could easily circumvent this problem by allowing students to look at the representation of the cube on paper, as well as an actual wooden cube. This would help all students access the math problem, as well as develop the cognitive skills necessary to associate a cube to a 2-D representation of that solid. An article by DeLoach, Scudder, and Uttal (1997) demonstrates how abstract spatial representations vary in appropriateness depending on the age and developmental readiness of the students being shown that model. For example, two and a half year old children were only able to locate objects in a room 20% of the time after being showed the location of the object in a diorama that was meant to serve a spatial model of the room. On the other hand, three year old children were able to correctly interpret the spatial model and find the hidden object 80% of the time. In this study, only six months of developmental difference lead to a significant change in spatial ability in these students (1997, p. 41). While a six month age difference would probably not lead to such a pronounced discrepancy in performance in high school students, these results suggest that developmental factors are important to consider when working with children and adolescents.

Similarly, in an article by Kim, Kim, and Kwon (1997), the authors discuss the use of virtual reality to help students improve their spatial sense, particularly as it relates to understanding
three-dimensional objects. They claim that this gives students an experience that is as close to having a tangible object to manipulate as possible; in fact, they even “…transcend the constraints of temporal access and physical space,” and also help students grasp how two-dimensional space can be used to describe three-dimensional objects (2002). In a similar context, Moyer, Bolyard, and Spikell (2002) also write about using “virtual manipulatives” in the classroom. These authors are careful to distinguish between virtual manipulatives that use static versus dynamic “visual representations of concrete manipulatives.” Namely, static representations could be described as computer image files, particularly ones that students cannot easily rotate, zoom, or modify (2002, p. 372). On the other hand, dynamic representations allow more interaction, such as a three dimensional solid that a student can pick the size of, rotate in space, or modify. Essentially, a manipulative goes from being static to dynamic as it becomes a better representation of an object a student might be able to interact with in the real world. But one of the main benefits of a virtual manipulative is that it can actually become even more interactive and adaptable than an object a student might encounter in the real world. While a student might benefit from holding an actual wooden cube when working on a unit on solid geometry, a student who is developmentally ready to interact with a virtual model can benefit from the fact that a virtual cube can be easily resized, recolored, or sliced into pieces (2002, p. 373).

While technology can be used to bring manipulatives into the classroom, Battista, Clements, Sarama, and Swaminathan (1997) showed that the important part of using manipulatives in the classroom is not whether they are real or virtual, but that they are used with the hopes of achieving certain mathematical goals. Among these include understanding geometric notions such
as distance and area, physics ideas such as velocity and acceleration, and connective notions about
the relationship between “spatial and numeric ideas” (1997, p. 172). In this study, the authors
showed that the third graders they studied were able to improve their spatial visualization abilities
by completing tasks related to shapes, area, multiplication, and fractions in computer programs and
by the guided use of physical manipulatives (1997, p. 173). The results of this study are promising
insofar as these tasks helped improve the students’ “spatial competencies,” but the authors did not
attempt to discover whether this improvement in spatial ability had a direct affect on their
mathematical ability or performance. Instead, they simply cited other sources that discuss the
strong kinship between spatial and mathematical abilities to imply that “the curriculum and the
software described here may contribute to students' mathematical development” (1997, p. 182).
Hence, as Moyer describes in her article, developing spatial skills to improve students' math ability
by using manipulatives and other techniques in the classroom is actually a quite complicated
process. As she says,

    Mathematical relationships must be imposed on the materials. The student's own internal
    representation of ideas must somehow connect with the external representation or
    manipulative. However, manipulatives are externally generated as manufacturers’
    representations of mathematical ideas; therefore, meaning attached to the manipulatives by
    manufacturers is not necessarily transparent to teachers and students. (2001, p. 192).

In essence, these tools have to induce a student's ability to construct “schematic imagery” rather
than simply assisting “visual imagery”: manipulatives and spatial reasoning skills are only useful
insomuch as the student can relate and apply them to the mathematical task at hand (Van Garderen,
Part 5: Summary

While a review of the literature may not reveal many answers to the questions posed in this research project, it is apparent that a large amount of academic work has been conducted in this subject. For example, despite only being a small part of my master’s project question, problem solving has been written about extensively in the literature. In particular, the process of problem solving can be broken down into its affective and cognitive aspects, both of which influence a student’s overall problem solving ability. Even within these subdomains of the domain of problem solving, there are subdivisions to be made, such as how metacognitive problem solving skills exist as a distinct subset of cognitive problem solving skills.

The study of spatial skills is equally as rich and complicated of a subject. Researchers in this domain have demonstrated distinctions between different kinds of spatial imagery, visual imagery, and the usefulness of each in problem solving activities, especially as they relate to mathematics. However, the specific relationship between spatial skills and raw mathematical ability has remained elusive, and many researchers have discussed that this is a difficult topic to study given current methods of collecting data, which really just focus on the connection between spatial ability and performance on carefully selected problem solving tasks.

Finally, much work has been done on how to improve spatial reasoning skills in students by using exercises, manipulatives or other activities. While other researchers have been able to determine what characteristics are necessary for making a manipulative a good spatial model for students to work with (physical manipulatives or dynamic virtual manipulatives), there are still some questions about how educators can help to connect mathematical ideas to the use of the
manipulative. That is, while manipulatives have been used to improve spatial ability in students, there is still much work to be done on the issue of how to connect that improved spatial ability to problem solving skills that can be applied on a mathematical task.
Data Collection

I began the process of collecting data by first getting permission from my mentor teacher at my student teaching site, the department chair in the math department at this school, and the high school principal at the school. I informally received permission to go ahead with this research project both in verbal conversations and through email correspondences. Then, I proceeded to secure written permission from the school principal and department chair. After this, I visited the three precalculus classes that I planned to study in order to obtain permission from the students to conduct focus group interviews.

Since my mentor teacher saved every single test her students took during the entire school year, including the tests that I administered during my student teaching internship, this served as a plentiful source of data that helped me to study my master's project question. I analyzed particular samples of these artifacts for evidence of students using spatial reasoning skills and how it affected their problem solving experience. I chose to analyze samples of work that the students completed when I was teaching them both because the material was more conducive to producing evidence of spatial reasoning and because I knew the material on the tests well enough that I knew exactly which problems to look for in order to pick work that would be relevant to my research.

Of course, these student solutions are only one source of data that I collected on this topic. I also conducted focus group interviews and administered surveys to students on these problem solving experiences to get a fuller picture of how they solved the problems I studied. The specific problems from my student teaching experience that I picked were constructed in such a way that required the students to use spatial and visualization skills, so these were the perfect problems to
analyze for certain traits such as the accuracy of the diagram included, the student's ability to use this diagram, and the final solution to the problem. When I interviewed several students about their solutions, I had an opportunity to learn more about their thought process when solving the problem. During this same focus group interview, I gave the students an opportunity to solve the spatial part of the problem again in a relaxed environment. I also asked them to explain their steps to me as they completed the problem again.

During this focus group discussion, I asked the students to talk about how they solve problems in general, how they feel like they should solve problems, what they picture in their minds when they have a “good” problem solving experience, and what they picture in their minds when they have a “bad” problem solving experience. I also asked them what they picture in their minds when they solve a problem they consider easy and what they picture in their minds when they solve a problem they view as hard. Finally, I administered a student survey that asked the students questions about how they solve problems and how they use diagrams when solving math problems. To summarize, the data I collected is as follows:

1. Analysis of student work (from tests throughout the school year)
2. Focus group interviews
   - Analysis of a group of students solving a problem and explaining his or her thought process
   - Feedback about student thoughts on problem solving, spatial skills, and what makes a problem easy or hard
3. Student survey about problem solving
I believe that these pieces of data have been extremely valuable in answering my research question. This was an opportunity to see how students use spatial skills when solving problems, whether those skills help them during the problem-solving process, and how spatial skills play into the problem-solving process depending on how easy or hard they feel the task they are completing is. Hence, by examining student work, interacting with students directly, and analyzing their thought process when solving problems, I think I got a holistic view of my research question.

When I conducted this research, I had to be careful to consider privacy concerns. While hundreds of tests from throughout the school year are a valuable resource, it was also important to make sure I used them in a way that was anonymous and could not negatively affect any of the students I studied. All tests I reproduced were carefully examined so that I could remove any information that could identify the students who completed one of those tests. Finally, during focus group meetings, I had to stress to the students that they could leave at any time if they felt uncomfortable sharing their thoughts or talking about how they solve problems. Social situations such as a group interview can be stressful for teenagers, especially if they feel like they are obligated to participate and share personal information. The survey was administered in the same way—the students were told that their participation was optional, and that their choice not to participate would not affect their grade in any way.
Data Analysis and Interpretation

Part 1: Analysis and Interpretation of Student Survey

For this section of my data analysis and interpretation, I will address an anonymous survey that I gave to three sections of precalculus students at a college preparatory in a city in the Midwestern United States. Out of a total of 69 students in these three classes, 63 students responded to the survey—a response rate of 91%. I administered the survey at the beginning of class for each of the three sections surveyed. The survey had a total of eight questions, which all related to problem solving and spatial skills. In order to encourage students to put as much thought as possible into answering each question, the eight questions on the survey were kept as concise as possible: each question consisted of a short statement, and the respondent was asked to record whether they strongly agreed with the statement, somewhat agreed, neither agreed nor disagreed, somewhat disagreed, or strongly disagreed.2 Because these statements were chosen for their brevity, all participants completed their surveys within the first five minutes of class time.

The first question on the survey asked the students whether or not they agreed with the statement “When I solve math problems, I try to imagine a picture related to the problem in my head.” Approximately 74.6% of students (47 out of 63) either strongly or somewhat agreed with this statement. An additional 12.6% neither agreed nor disagreed with the statement (8 out of 63), and the final 12.6% of students either somewhat disagreed or strongly disagreed with this statement. From these responses, it seems that a large proportion of the students surveyed try to use mental pictures when solving math problems.

2 See Appendix I for a blank sample of this survey.
The second question on the survey was “The better I can imagine a picture related to a math problem in my mind, the easier it is for me to solve it.” Approximately 82.5% of respondents (52 out of 63) either strongly or somewhat agreed with this statement. An additional 11% (7 out of 63) neither agreed nor disagreed with this statement. Finally, 6% of respondents (4 out of 63) either somewhat or strongly disagreed with this statement. While this result cannot be safely generalized beyond this particular sampling of high-achieving students, it is reasonable to assert that a significant portion of students who responded to this survey felt that there is a correlation between their ability to imagine a picture related to a math problem and their ability to solve that problem.

The third question on the survey was “When I solve a math problem, the first thing I like to do before writing anything down is to picture the problem in my mind.” Approximately 50.8% of students (32 out of 63) agreed with this statement to some extent. An additional 23.8% (15 out of 63) neither agreed nor disagreed, and the remaining 25.4% (16 out of 63) disagreed with the statement to some extent. This result shows that in this sample of students, over 50% students who responded to the survey felt that it is valuable to picture math problems in their minds before writing anything down on paper. However, almost as many students were either indifferent or disagreeable towards visualizing a problem before writing anything down when solving a math problem.

The fourth question on the survey was “I draw pictures or diagrams to help me solve math problems whenever possible.” The majority of the respondents, 87% or 55 out of 63, reported that they either agreed or strongly agreed with this statement. A further 7.9% (5 out of 63) neither
agreed nor disagreed with the statement. Finally, the remaining 4.8% of students surveyed (3 out of 63) reported that they either somewhat disagreed or strongly disagreed with this statement. From this question, it seems reasonable to infer that most of the students in the sample are agreeable towards drawing pictures or diagrams when solving math problems.

The fifth question on the survey was “When I draw pictures or diagrams to help me solve a math problem, I try to draw it to scale.” A slim majority of students, 52.4% or 33 out of 63, reported that they either strongly or somewhat agreed with this statement. A further 15.9% (10 out of 63) neither agreed nor disagreed with this statement. Finally, 31.7% (20 out of 63) disagreed with this statement to some extent. While over half of the students surveyed reported that they try to draw pictures and diagrams to scale in a math class, almost as many students did not make this a priority—approximately 16% of students reported indifference towards this practice, while the remaining 32% of students reported that they do not do this. In this particular sampling of students, it seems that there is an almost even split between those who draw diagrams to scale and those who do not.

The sixth question on the survey was “When I draw pictures or diagrams to help me solve a math problem, I try to label important aspects of the diagram such as angles, distances, or heights.” A large majority of students, 95.2% or 60 out of 63, reported that they labeled diagrams or pictures when solving math problems. One student, 1.6% of the sample, reported neither agreeing nor disagreeing with the above statement, and two students, 3.2% of the sample, reported that they do not use pictures or diagrams when solving math problems. More students reported that they strongly agreed with statement than on any other question on this survey (66.7% of respondents, or
42 out of 63). In this case, almost every student in these three precalculus classes reported that they try to label their diagrams with important data such as angles, distances, or heights. This result may be traceable to the fact that all students surveyed attended the same high school, and all had the same precalculus teacher. It is likely that this survey question received more affirmative responses than might be reported in a representative sample of precalculus students—this is because labeling diagrams might be stressed more in the high school math classes I studied than in other schools or perhaps it is made a higher priority by the precalculus teacher from which these students all learned.

The seventh question on the survey was “I find it easiest to do a word problem in a math class if I translate the words into a picture or a diagram.” A sizable majority of student surveyed, 76.2% or 48 out of 63, somewhat or strongly agreed with this statement. 19.0% of respondents (12 out of 63) neither agreed nor disagreed with this statement, while only 4.8% (3 out of 63) disagreed with this statement to some extent. This result indicates that a large majority of students surveyed associated the ease at which they can solve a word problem with whether they can make a diagram encoding the situation described in that problem.

The eighth question on the survey was “When working on a word problem in a math class, I find it hard to convert the words into a picture. However, once the picture is done, then the rest is easy.” A small majority of students, 50.8% or 32 out 63, agreed with this statement to some extent. An additional 27.0% (17 out of 63) neither agreed nor disagreed with the statement. Finally, 22.2% (14 out of 63) disagreed with this statement to some extent. While 76.2% of students said in the previous question that they find it easiest to solve world problems if they translate the words into a
picture or a diagram, this question revealed that the converse of this statement was not as likely to hold true for the sample I surveyed—only 50.8% of the sample agreed that once they convert a word problem to a picture, the rest of the problem is easy. This survey question also revealed that almost 50% of the sample feel that converting a word problem into a meaningful diagram is a challenging process. This is an important finding because it confirms that these students view utilizing spatial and visual skills in the context of a math class both as something that is an important part of the problem solving process and as something that is challenging.

One critical result of this survey was that the majority of the students surveyed agreed with all of the statements on this survey to some extent. Hence, it seems reasonable to make the generalization that the majority of the students who responded to this survey used mental pictures to help them solve math problems, and also used charts, pictures, or diagrams to help them make this visualization concrete.

In addition to analyzing the frequency of certain responses to individual questions on this survey, there are also meaning correlations between certain responses on the survey that are worth analyzing. For example, 38.1% of respondents (24 of 63) agreed with the first and fifth questions on the survey, meaning that they try to imagine a picture related to a math problem when trying to solve it and also try to keep any diagrams they draw to scale. Further, 40.0% of respondents (25 of 63) agreed with the first question on the survey but disagreed with or were indifferent to the fifth. 7.9% of respondents or 5 of 63 disagreed with or were indifferent to both statements. Finally, 14.3% or 9 of 63 respondents disagreed with or was indifferent to the first question, but agreed with the fifth. These results can be seen graphically in the following figure:
Total number of student included in sample: 63

We can begin by interpreting those who agreed with the first survey question as visual problem solvers, while those who disagreed with or were indifferent to this question can be classified as non-visual problem solvers. Then, among the visual problem solvers, we can see that there exists a nearly even split between those who draw diagrams to scale and those who do not. Van Garderen (2006) notes that there exists an important distinction among those who use visualization skills to solve math problems—there are those who use “schematic imagery” and those who use “visual imagery” (p. 497). Schematic problem solvers encode spatial relationships in their visualizations and diagrams, while those who only use visual imagery do not. These survey results seem to confirm this notion that there exists such a distinction. However, van Garderen
proceeds to claim that schematic imagery is “more sophisticated” and that it is an indicator of problem solving success in students studying math (p. 497). This claim is beyond the reach of the results of this survey—the data collected on this survey cannot be interpreted in a manner than can either support or refute the claim that those who utilize schematic imagery are more likely to succeed in problem solving tasks.

There is another significant correlation involving the fifth survey question regarding drawing diagrams to scale. In this case, 94% of respondents who agreed with the statement in survey question five agreed with or were indifferent to the statement in the eighth survey question, “When working on a word problem in math class, I find it hard to convert the words into a picture. However, once the picture is done, then the rest is easy.” In addition, 57.6% of respondents who agreed with the sixth survey question strictly agreed with this statement. This is significant because among those who disagreed with the statement in question six or were indifferent towards it, only 66.7% of respondents agreed with or were indifferent towards the statement in question eight. In addition, among those who disagreed with the statement in question six or were indifferent towards it, only 46.7% strictly agreed with the statement in question eight. Hence, we see that there appears to be a correlation between trying to draw diagrams to scale and how easy students believe it is to solve a math problem once a diagram is drawn.

Of course, it is important to reiterate that this analysis was done using a relatively small sample size and is based on survey results that might have been influenced by the structure of the survey. Certain respondents who agreed with most statements on the survey could have agreed for several reasons. One would hope they agreed because they actually felt this way when taking the
survey. However, the students might have felt like I, the researcher, was hoping for them to agree, and, in an effort to please me, proceeded to do so. Finally, it is also possible that once for students who agreed with the first few statements on the survey, they may have slowly become more inclined to agree with later items on the survey either out of habit or because they began to read later statements with the expectation that they would probably agree with them. Despite these potential shortcomings with this survey, it does seem likely to reason that these results support van Garderen's claim that there exist two distinct kinds of visualization techniques—schematic imagery and visual imagery. Further, while these survey results cannot support van Garderen's additional claim that schematic imagery is a more sophisticated kind of imagery and that students who use it are more likely to solve problems correctly, these survey results do indicate that students surveyed who use schematic imagery when solving problems were more likely to respond that once they successfully draw a picture or diagram representing a math problem, then the rest of the process is easy. That is, it seems that using schematic imagery, if nothing else, helped these students feel confident that once they drew a picture of the problem situation, then it would be easy for them to successfully solve the problem at hand.

One way that I could have improved this survey in order to obtain even more data would have been to ask students to anonymously report demographic information such as their age, sex, and year in school. This would have allowed me to determine whether reported use of spatial or visualization skills was correlated with a student's age or year in school. Likewise, these survey responses could have been correlated with performance in math classes had I asked students to report the quarter grades they received throughout the year, though this question could have had
ethical questions regarding the anonymity of the survey.

**Part 2: Analysis and Interpretation of Student Work**

In this section, I will analyze student work that I collected from the same three sections of precalculus students that I worked with in the last section. The student work is from a test that I administered to these students during my student teaching internship. This test assessed their knowledge of trigonometry starting from basic definitions through proving and using trigonometric identities. Out of a total of 72 students in these three classes at this time, 71 completed the test. In addition, I removed three tests from the sample because these tests were redone at home or with a tutor after school hours. Hence, 68 out of 72 students (94.4% of possible students) were included in this sample. The first question I analyzed required students to use numerous mathematical problem solving techniques. Firstly, the students had to translate an intricate word problem into a diagram. Secondly, they had to reason that since trigonometry is the study of triangles, they needed to find a way to incorporate triangles into this diagram. Next, they had to devise an algebraic setup using trigonometric functions that encapsulated the data in the diagram. Finally, they had to use this algebraic setup to get an answer to the problem. This problem read as follows:

1) Suppose that you are $x$ feet away from a building looking at a rectangular billboard on the wall of a building. The bottom of the billboard is 15 feet above the ground, and the billboard is 6 feet tall from bottom to top. Let $\theta$ be the angle (in radians) between your line of sight to the bottom of the billboard and your line of sight to the top of the billboard, assuming that your eyes are at a height of 5 feet. Find a formula for $\theta$ as a function of $x$.

Though there is no single way to represent this information accurately in a diagram, a typical correct diagram would resemble the one in the following figure:
In order to analyze the samples of student work that I collected, I first assessed the main characteristics necessary for creating an accurate diagram. I also considered other key criteria that related to my research. I then recorded whether these elements were present or not for each test in the sample. In total, there were ten elements of this problem that I considered.

The first element that I studied was whether or not the overall diagram was represented correctly. I determined that a diagram was represented correctly if it did not contain any inaccuracies. A lack of a diagram altogether was classified as an inaccuracy. While this is not an inaccuracy in the strictest sense, it is extremely unlikely that someone could solve this problem without a geometric interpretation of the question. In addition, the students in this class were
taught that a diagram will be a part of their grade on a trigonometry word problem. I might have reconsidered this classification had there been any students who got a correct solution to this problem without a diagram, but this situation did not arise in the samples of work that I analyzed. It is important to note that I still marked a diagram as correct even if it left out key information necessary for solving the problem. For example, if a student had left out the dotted line above the label “x feet” in Figure 2, I still marked the diagram as correct because even though it was missing a feature necessary to solving the problem, it did not, strictly speaking, contain any inaccuracies. However, had the student labeled any other angle in the diagram as \( \theta \), then I would record this as an incorrect solution. Approximately 73.5\% (50 out of 68) students in the sample included a correct diagram.

The second element I looked for when analyzing this pieces of student work was the presence of an indication that the line of sight to the billboard started at the eye level of an observer at precisely five feet off the ground. I marked this element as present as long as the line of sight of the observer started at five feet off the ground, or if the diagram was simplified in a manner that while the five foot observer was not included in the diagram, this presence was assumed. I determined this element of the diagram to be absent if the line of sight to the billboard began at ground level. In this case, approximately 88.2\% (60 out of 68) students included this as an element of their diagram. Students had to meet this criterion in order to have a correct diagram, and in order to solve the problem correctly.

Thirdly, I looked to see whether students had drawn in a horizontal line at the height of five feet in order to create two new triangles, both with a base of length \( x \) feet. These triangles can be
seen in Figure 2, with a base of a dotted line. In this case, exactly 75% (51 out of 68) students had this attribute in their diagrams. This element of the diagram was important because it allowed students to utilize the geometry of right triangles in order to help them solve the problem.

The fourth attribute of the students’ diagrams that I looked for was whether the students correctly labeled their diagrams or not. If they mislabeled a length in the diagram, failed to label a crucial length, or put the angle $\theta$ in the incorrect location, then I recorded them as having an incorrectly labeled diagram. Otherwise, I marked it as having correct labels. In this case, 75% of students (51 of 68) labeled their diagrams correctly. I should note that after collecting data for this measure, I realized that it would be logical to assume that it would yield the same exact results as the first measure (whether the overall diagram was correct or not). However, 50 out of 68 students had a correct diagram while 51 out of 68 had correct labels. This occurred because while it was necessary to have correct labels in order to have a correct diagram, it was not necessary to have a correct diagram in order to have correct labels. This happened for exactly one student—in all other cases, these measures yielded the same result.

The fifth attribute I looked for in these students' diagrams was whether they had correctly included a line of sight from some point below the billboard to the top of the billboard, and another line of sight from this same point to the bottom of the billboard. In this case, 76.5% of students (52 out of 68) successfully integrated this line of sight into their diagrams.

The sixth element I looked for in the diagrams the students' diagrams was a labeling of the billboard height as 6 feet tall. Students who either did not explicitly label the height of the billboard or mislabeled it were said to have labeled its height incorrectly. In this case, 83.4% of
students sampled (57 out of 68) labeled the height of the billboard in their diagrams.

Next, I looked at these diagrams in reference to van Garderen's (2006) distinction between “schematic imagery” and those who use “visual imagery” (p. 497). Schematic problem solvers encode spatial relationships in their visualizations and diagrams, while those who only use visual imagery do not. In addition, those who use visual imagery are more likely to concentrate on inconsequential details in their mental pictures rather than more important spatial data. For example, if one student who uses schematic imagery and another who uses visual imagery are both asked to imagine a round swimming pool that is 40 feet in diameter that they are 20 feet away from, then the two students are likely to have very different pictures in their minds. The schematic thinker would be more likely to imagine the diameter of the pool being twice as long as it is far away from the observer, while the visual thinker is more likely to think about details such as the color of the swimming pool liner or the kinds of toys floating in the pool. Hence, the seventh attribute I looked at was whether students drew their diagram roughly to scale. I measured this by seeing if the five foot height of the person in the problem was slightly shorter than the 6 foot billboard, and I checked if the 10 foot distance between the person's line of sight to the bottom of the billboard was approximately twice as long as the 5 foot height of the person in the problem. I treated this as an indicator of whether or not they used schematic imagery when solving this problem, since a written diagram is the closest approximation one could expect to get of a mental picture used by a student. In this case, exactly 50% of students in the sample (34 out of 68) had diagrams that were drawn roughly to scale.

In this same manner, the eighth attribute I looked for was whether students had pictorial
elements to their diagram—usually a stick figure or a detailed drawing of a billboard. Since this was visual data that was unrelated to solving the ultimate math problem, I used this as an indicator of visual imagery. For this measure, 52.9% of students sampled (36 out of 68) had pictorial elements in their diagrams.

The ninth attribute of the students' diagrams that I studied was whether they had a correct algebraic setup for solving the problem—typically this involved invoking the Pythagorean theorem or using inverse trigonometric functions to find the lengths of the observer's line of sight in terms of the variable \( x \). While the previous elements were included in the majority of the students' diagrams, a correct algebraic setup for this problem was much less prevalent for the students sampled. Only 27.9% of students (19 out of 68) used their diagrams to successfully set up this problem.

Finally, the tenth measure was whether students got the right answer to the problem in the end. Only 19.1% of students (13 out of 68) actually put their diagram and algebra together in a way that led to them arriving at the correct answer to the problem. Part of the reason that these last two attributes of the students' work were less common may have been that in order to complete the later steps of the problem (such as setting up the problem or arriving at the final answer), the students needed to have successfully completed almost all the earlier steps in the problem. For example, students who did not draw in a dotted horizontal line at five feet, such as in Figure 2, could not have easily arrived at the answer to this problem. Hence, by the later steps of this problem, not many students remained “mistake-free” and still in a position to arrive at a correct answer. In addition, some students may have had a difficult time transferring from a visual model of the
problem in the form of a diagram to an algebraic interpretation of the situation.

By comparing students who drew the diagram to scale to those who got the correct answer on the problem, we see that approximately 5.8% of students (4 out of 68) had a diagram that was not to scale yet had a correct answer, while 44.1% (30 out of 68) had a diagram that was not to scale and got an incorrect answer. Next, 36.8% of students (25 out of 68) had their diagram to scale but had the incorrect answer. Finally, 13.2% (9 out of 68 students) had their diagram to scale and had the correct answer. For a graphical representation of these data, refer to Figure 3. Of the students who got the problem correct, 9 drew the diagram to scale, while 4 did not. So in this case, over twice as many of those who got the answer correct drew their diagram to scale. In addition, of 34 students who drew their diagrams to scale, 9 got the answer correct—a success rate of 26.5%. On the other hand, of the 34 students who did not draw their diagrams to scale, only 4 got the problem right—a success rate of 11.8%.
Next, by comparing students who used pictorial imagery to those who got the correct answer, we see that 5.8% of student (4 out of 68) had a diagram that incorporated pictorial imagery, yet still had the correct answer, while 47.1% (32 out of 68) had diagrams with pictorial imagery and got an incorrect answer to the problem. A further 33.8% of students in the sample (23 out of 68) had a diagram that did not incorporate pictorial imagery, but did not have the correct answer. Finally, 13.2% of students (9 out 68) had a diagram with no pictorial imagery and also had the correct answer to the problem. See Figure 4 for a graph of this data. So in this data set, a student who got the answer right was over twice as likely to have left out pictorial imagery than they were to have used it (of 13 students who go the problem correct, 9 did not use pictorial
imagery, while 4 did). In addition, 28.1% of students (9 out of 32) who did not use pictorial imagery obtained the correct answer on the problem. On the other hand, only 11.1% of students (4 out of 36) who used pictorial imagery obtained the correct answer to the problem.

**Figure 4:**

![Pie chart showing the relationship between pictorial imagery and correct test answers.]

*Total number of students included in sample: 68*

It is crucial to note that while these correlations exist in the data, the sample size in this study was too small to infer any conclusive findings. However, these findings still support van Garderen's (2006) claim that students who use schematic imagery are more likely to successfully solve math problems than those who do not use schematic imagery, particularly those who use pictorial imagery. We see that a subset of 9 of the 13 students who solved the problem correctly used diagrams that were drawn to scale (evidence of using schematic imagery when solving the...
problem), while another subset of 9 of the 13 students who solved the problem correctly left out pictorial details in their diagram (evidence of not using visual imagery when solving the problem). Note that drawing a diagram to scale and including pictorial details in a diagram were not mutually exclusive attributes. With the measure I used, it was possible to have a diagram that was drawn to scale and incorporated pictorial details. In fact, 3 of the 13 students who solved the problem correctly had both attributes in their diagrams. Another 3 of the 13 students had neither attribute in their diagrams. 1 of the 13 students had a diagram that was not drawn to scale but had pictorial details. Finally, the remaining 6 of the 13 students who solved the problem correctly had a diagram that was drawn to scale, but did not incorporate pictorial attributes.

Although the student survey and this analysis of student work addressed different angles of the question about how spatial skills relate to mathematical problem solving ability, there is evidence of convergence between these two data sources. In the student survey, 94% of respondents who said that they try to draw diagrams to scale agreed with or were indifferent to the statement in the eighth survey question, “When working on a word problem in math class, I find it hard to convert the words into a picture. However, once the picture is done, then the rest is easy.” This supports the results in the analysis of student work, where we found that among the students sampled, those who drew spatially accurate diagrams on the trigonometry problem discussed above were more likely to solve it correctly. While neither the correlation in the student survey nor the correlation in the analysis of the student work can be claimed to be causal links, it does appear that in both cases, schematic imagery was associated with solving math problems correctly.
Part 3: Analysis and Interpretation of Focus Group Discussions

The final data source that I will discuss is a pair of focus group discussions that I conducted with five students selected from the precalculus classes I worked with for my student survey and analysis of student work. I recruited students for this focus group by visiting each of these three sections of precalculus at the beginning of their class period and telling them about my research project and my need to conduct a focus group interview. I handed out release forms to all students in these three classes and told them the date and time that the interview would take place. Approximately eight students originally agreed to participate, but only five actually attended the interview. Three students came at the time announced, so a first focus group was conducted with these students. An additional two students arrived almost an hour late for the focus group discussion because their schedule was changed due to Advanced Placement examinations, so I conducted a second, separate discussion with these students. Because of how I recruited students for this discussion, students were self-selected as participants. Though this is an anecdotal generalization based on my previous experience working with these particular precalculus classes, the students who chose to participate in the focus group discussion were the students that I would characterize as being among the highest achievers in their math classes. This fact, combined with the fact that I conducted my research at a selective urban magnet school, means that the results of this focus group discussion are likely skewed towards the opinions of high-ability, high-achieving students.

Because of the priorities of the particular students that I interviewed, the discussion seemed to focus around the study of mathematics as a means to achieving a high grade on a report card. As
a result, students often brought up the notion of memorizing formulas or attaining procedural knowledge as a means of solving mathematical problems. For example, when I asked the students to describe in as much detail as possible about the process they go through when solving math problems, one student had the following response:

Jon: I try and like, I guess, identify a formula. You know, like, you want something where you can put it into where you can start solving. So I guess a formula is the best place to start.

Even when asked later whether he uses mental pictures to help him solve math problems, this student said that “I guess I'm much more formula based. I like following a formula and working through it.” So despite my effort to determine how these student use visual and schematic imagery when solving math problems, this student kept returning to the idea of mathematics as a subject that requires one to memorize a formula and to just plug entries in to obtain an answer. Later in the interview, when asking students whether they preferred to work in their minds or on paper when solving problems, one student framed this same idea in a different way.

Maria: Same here [indicating a preference for solving problems on paper]. I always, like, am, like, doing problems regardless of if they, like, taught us. I always like to have my textbook so I can see it, like an example that is pretty much the same thing.

In this case, the student is indicating her preference for solving problems by imitating nearly identical examples that can be found in her textbook. In part, this indicates that the student views the “cognitive aspects” of problem solving as being important—that is, a lot of problem solving ability comes from building a strong repertoire of past experiences that one can refer back to when solving new problems. Indeed, Boekaerts, Seegers, and Vermeer (1995, p. 241) discuss the importance of past experiences in solving new problems. However, when Maria says that she likes
to find an example that “is pretty much the same thing,” it seems that she is also partially discussing her desire to copy a procedure from a textbook without fully understanding the concepts behind that procedure. Jon seems to support this assertion:

    Jon: If something is really hard to get, and like I said, going back to the conceptual part, it's something I'm having a hard time understanding, and I know like lots of other people do too, you do more like memorization than actually learning how to do something. Like you'll just memorize how somebody did the formula without understanding how or why someone did the formula. And sometimes it works and sometimes it doesn't. But I know I do that, and I know a lot of my friends do it too.

In this response to a question about what the students would most like to improve with regards to their mathematical ability, Jon indicates that he would like to do a better job of learning the conceptual side of math rather than simply resorting to rote memorization.

    However, many of these students did not seem to talk about this in reference to improving their learning in a math class, but rather in reference to improving their grades. As Letitia says in the second focus group discussion, she “keep[s] her grades very much in mind.” Later in the interview, Jon seems to support this idea:

    Jon: And then there's also how sometimes teachers like to throw curveballs on tests. I mean like, they don't bring in something completely that you haven't done before, but like they really test your comprehension of the subject matter by giving you a question you haven't seen before. But it's like dealing with similar stuff. Those, those throw me off.

In essence, Jon notes that he finds test questions difficult when they are similar to problems he has completed in the past, but not exactly the same. So the consequences of his habit of memorizing formulas instead of learning conceptual information are not related to his overall learning, but his performance on a test.

    However, students also cited using other problem solving techniques referenced in much of
the literature about the topic. Both Jon and Maria noted that they used “metacognitive skills,” as defined by Boekaerts, Seegers and Vermeer (1995, p. 242). Namely, they talked about how they felt it was important to, as Jon said, “picture your plan” before solving a problem and think about the “steps you're gonna use.” Maria also talked about the importance of taking an organized approach to problem solving:

Maria: You have to be organized too. 'Cause otherwise you have to go back. You should, like, plan it out in steps and lines and not, like, scribble everywhere.

Though the questions I asked during the focus group discussion were meant to direct the dialog towards a discussion of visualization and diagrams, the first time the students discussed this topic was in the context of using diagrams as an organizational tool. Maria noted that when she uses diagrams, “it's usually for longer problems” so she could keep track of her work. At another point in the discussion, Karen supports this idea when she says that “There's some cases where like there's a lot of information, I'll try to diagram it out to see, to see if I can visualize it better.” In this sense, diagrams can be used to stay organized when students work on an intricate math problem that requires them to keep track of a lot of information at once. Karen also notes that diagrams serve as an aid that helps her visualize the problem.

One interesting finding from this focus group discussion was that even though the students regularly steered the discussion away from spatial skills and visualization, all of the students interviewed described themselves as “visual learners” when asked. There seemed to be somewhat of a disconnect between this assertion and the fact that they often cited using problem solving techniques that did not related to spatial or visual imagery. Even when asked questions that were designed to steer the discussion towards spatial or visual imagery, they still steered away from this
issue. From the topics discussed in these focus groups, it seems that these students have had extensive training and experience with how to successfully solve math problems. They discussed the necessity of organizing their thoughts, planning out the steps of the problem solving process, relating the problem to one they have done before, underlining keywords in the problem statement, and finding a mathematical formula related to the problem. However, these students did not seem to have much training in how to use visualization techniques or diagramming to help them solve math problems. While this research was not extensive to say this conclusively, it does seem that these students would have benefited from having a teacher explicitly cover the visual, spatial, and diagrammatic aspects of problem solving.

This conclusion comes partly from the topics that were and were not brought up during the focus group discussion, but it was also confirmed from the second half of the focus group discussion when the students worked through the same problem that I analyzed in the last section about student work. While the students participating in the focus group had seen this problem before, it had been approximately three months since they had covered that material. During this part of the focus group discussion, I asked a student to read the problem statement aloud. Then I asked the students to read the problem statement to themselves twice more. Then I asked them to describe what they had visualized after rereading the problem statement, but before writing anything down on paper. After completing this exercise, all of the students participating in the focus group discussion were able to successfully draw an accurate diagram of the situation that contained all important aspects of the problem as measured in the previous section in which I analyzed samples of student work. It seemed that the students benefited from this exercise and the
stress I put on describing how they visualized the problem, because they had a higher success rate with drawing this diagram than the general student did when I analyzed their test results. This higher rate of success could have also been related to the fact that this was a low-stress situation in which their work had no implication on their grade, there were no other problems to solve, and there was no set time limit on their work. It could have also been skewed by the fact that notably high-achieving students participated in the focus group or the fact that the students had seen this problem before. Nonetheless, the results of this focus group discussion offer promising preliminary results that invite further research and inquiry into whether specifically stressing visual, spatial, and diagrammatic skills would improve a student's ability to solve math problems.
Conclusions

The results of this project have several implications for other researchers in this field of inquiry. Firstly, as I discussed in my data analysis and interpretation, there is evidence of convergence between two of my data sources. In the student survey I administered, I found that there exists a correlation between students reporting that they used spatial reasoning when solving math problems and their perceived ability to solve math problems. Further, in the samples of student work that I analyzed, I found there exists a correlation between students exhibiting evidence of using spatial reasoning and whether they solved the math problem correctly. While these results are by no means definitive, they do help to support the hypothesis that there exists a positive correlation between a student's spatial ability and his or her mathematical problem solving ability. Hence, my research invites additional inquiry on the topic.

Likewise, findings from my individual data sources will also help future research in this field. The findings in my analysis and interpretation of samples of student work helped to confirm van Garderen's (2006) claim that the use of mental imagery that focuses on encoding spatial data is “more sophisticated” than imagery that concentrates on encoding visual information about the problem (p. 497). That is, my analysis of student work showed that students who drew spatially-accurate diagrams were more likely than other students to solve this particular problem correctly. By supporting this researcher's findings, my project helps to highlight the need for further inquiry into what kind of imagery students use when solving math problems and how this affects their subsequent solutions to those problems.

Finally, my data show that there exists a discrepancy between how students use diagrams to
solve math problems and how they describe the problem solving process. Approximately 73.5% of students sampled drew a correct diagram when trying to solve the math problem I focused on. Further, 87% of students surveyed reported using diagrams or pictures whenever possible when solving math problems. However, during the focus group discussions I held, students seemed to prioritize other problem solving strategies over using mental imagery or diagrams—this held true even when I asked them questions that were meant to elicit responses regarding how they used imagery or diagrams, and even though the students I studied were in a math class in which using visual problem solving techniques was arguably one of the most important techniques they could use in order to achieve success in the class. While these findings are in no way definitive considering the small, unrepresentative sample size in each of my data sources, this finding still has implications in the field because it is an interesting finding that others, including myself, may want to research further—first, by verifying that this phenomenon actually exists and, second, by trying to determine why this phenomenon might be occurring.

However, while my research allowed me to collect data that painted a colorful, rich picture of how students use spatial skills when solving math problems, there were several lessons that I learned from this process. First of all, I learned that one must not let lofty goals and overly ambitious plans limit one's research before it has even begun. That is to say, by drafting goals for a project that are unreasonable to complete with available time and resources, a researcher can severely limit how successful his or her research can be. As I described in my section on my “Research Question Development,” my original plans for this project could be described at best as slightly too ambitious—I wanted to perform a longitudinal study in which I worked on spatial
exercises with a group of students every day for a period of at least 5 weeks. I hoped to also have a control group of students that I could compare with the experimental group so that I could see which group showed greater gains in both spatial ability and mathematical problem solving ability. Instead of proposing a project that I knew to be too difficult to complete only to have to modify it later while trying as hard as possible to maintain its original integrity, I would have had more success had I considered the feasibility of my plans from the start.

In addition, I learned after collecting my data that my research would have benefited from adding another step between the data collection plan and the data analysis and interpretation. Namely, I learned that after drafting the data collection plan and creating my data collection materials such as an interview protocol and a student survey, I should have completed a “mock data analysis.” More specifically, I should have put my student survey, interview protocol, and tentative samples of student work together to determine (1) exactly how these data sources would work together to answer my research question and (2) how I wanted these data sources to work together to answer my research question. While I did consider each data source individually and tried to make sure it would be as effective a tool as possible to help answer my research question, I did not consider the importance of verifying that there would be some level of “synergy” between the data I collected from all three sources. For example, when drafting my student survey and deciding how I would collect data from samples of student work, I made sure to include a measure for whether students use spatial information when solving problems. However, I did not include any sort of specific measure for determining this in my interview protocol. My interview protocol was still designed to address my research question, but it was not designed in a way that allowed for me to
claim that there existed convergence between my findings from all three data sources. In summary, I learned that effective triangulation of data sources does not just involve picking three sources of data that will help to answer the same question, but rather it requires one to try to guarantee that these data sources can be compared and contrasted in as many meaningful ways as possible when trying to answer one's research question.

This lesson learned about how to design an effective data collection plan leads to one of the major limitations of this study—the data I collected in the course of this research was not collected in a way that allowed me to maximize the benefits of triangulation. That is, I was not always able to discuss a finding in one of my data sources in relation to my findings in my other two data sources. As a result, I created a situation in which my study, while still valuable, was fundamentally self-limited.

In addition, as I mentioned previously, the findings of this study are limited by my small sample size and the unrepresentative sample of students I studied. Each data source had a relatively small sample size: I received 63 responses to the student survey I distributed, I analyzed 68 samples of student work, and 5 students participated in focus group discussions that I led. This small number of students sampled simply does not allow one to make any sweeping generalizations or conclusions. In addition, the students I studied were all from the same city, attending the same selective magnet school, and were all taking the same math class. Hence, the students I studied were in no way a representative sample of the American high school student population. Because of this limitation, the results of my research have to exist within the confines of where and how my data was collected. The majority of my findings are conclusive only for the specific students I
studied in the specific context in which I studied them. All other generalizations or broader conclusions that I could make are only valid as possible conclusions that would need to be verified by further study.

In addition, the way in which I collected data combined with the relationship I had with the students I studied could be a limitation on my study. Since I performed my research in the same school in which I completed my student teaching internship with the same students I taught for three months, these students may have been more likely to modify their responses on the survey or during the focus group interview in order to give me the response for which they might have thought I was looking. For example, since the students knew I was studying how spatial skills affected how a person solved math problems, they might have been more likely to report on the survey that they used spatial skills or mental pictures because they thought that this is what I was hoping they would say. Similarly, the students might have chosen responses during the focus group discussion that they thought I was looking for, or at least given responses that took into account that they were having a discussion with their former teacher, and that there current teacher was in the room and could overhear the focus group discussion that I was facilitating.

Since there were several aspects of my study that limited my results, these imply possible changes to my data collection plan that I would consider were I to repeat this study. Firstly, I would modify my interview protocol by including questions that related more closely to the questions asked in the student survey regarding spatial skills and drawing diagrams to scale. Further, I would consider modifying how I collected my survey and student work data in a way that would allow me to compare an individual student's survey to his or her sample of work. The way I anonymously
collected data for this project resulted in my not being able to link a student's work to his or her survey. One way this could be achieved would be by attaching the collected samples of a particular student's work to a blank survey, writing the student's name on a adhesive “Post-it” paper, attaching it to the survey/sample, and then blacking out all evidence of the student's name on the sample of work. Then, I could hand out the survey/sample to the students according to the name on the “Post-it” paper and ask them to remove their name from the survey/sample before turning it back in to me. This way, the survey would remain anonymous, but I could still compare case-by-case results between these two data sources. Such a technique would have allowed me to maximize the amount of analysis I could have done across these two data sources. For example, I could have seen whether there was a strong correlation between students drawing their diagrams to scale and them reporting on the survey that they try to draw diagrams to scale whenever possible.

In addition, if I were to repeat this study again, I would have modified the anonymous survey I wrote in a way that would have allowed me to collect more demographic information about the students being surveyed. For example, it would have been possible to ask the students their gender or their age while still maintaining the anonymity of the survey. This would have allowed me to answer questions regarding whether my results varied based on gender or the age of the students studied. While these questions were not strictly related to my research question, this would have been valuable data to collect since it could have been developed in further studies or related to work by other researchers who do look at the effects of gender or age on how students use spatial skills when solving math problems.

While the results of my research do not justify the universal use of a new model for
teaching problem solving or how to use spatial skills, my results imply to me that I should try to emphasize skills such as drawing diagrams or taking a minute to develop a mental picture of a problem when I work with students in the future. Despite the fact that I cannot definitively say whether this will help students to solve math problems, my results do imply to me that this is worth testing further. Testing out this model of emphasizing diagrams and mental pictures is a way for me to continue my research into my educational career in a way that I believe, though cannot yet prove, will benefit my students.

One major suggestion for further research in this field would be to test whether my results can be generalized to more students. This would involve conducting a study with a significantly larger sample that incorporates the changes to the data collection plan discussed above. This further research would help to verify whether or not my results can be generalized and used to develop a new model for teaching problem solving in a general math classroom.

In addition, the results of my research indicate to me that it would be valuable to continue this work by going back to the original research question that I began with when I started this project. As I discussed in my Research Question Development section, I originally started this project with the intent of studying whether trying to develop students' spatial skills would improve their math ability. At this point in my research, it seems that this question has become relevant again because I was able to show that, at least in the students I studied, using spatial skills was associated with solving math problems correctly. Hence, it would be logical to test whether trying to develop spatial skills in students by having them complete daily spatial exercises would (1) actually improve their spatial ability and (2) have a positive affect on their problem-solving ability.
However, unlike my plan for this original research question, I would recommend that these daily spatial exercises be related to the mathematical content the students are learning in class so that there can be an effort not only to develop a student's spatial reasoning ability, but to relate that reasoning ability to the context of mathematical problem solving.
References


Appendix I: Student Survey Template
Problem-Solving Survey

Below are some statements related to problem solving. Please read these statements and record to what extent you agree with them by circling a number below the statement.

1) When I solve math problems, I try to imagine a picture related to the problem in my head.

<table>
<thead>
<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
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</table>

2) The better I can imagine a picture related to a math problem in my mind, the easier it is for me to solve it.

<table>
<thead>
<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
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<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
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</table>

3) When I solve a math problem, the first thing I like to do before writing anything down is to picture the problem in my mind.

<table>
<thead>
<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
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</table>

4) I draw pictures or diagrams that help me solve math problems whenever possible.

<table>
<thead>
<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
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<tr>
<td>2</td>
<td>1</td>
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5) When I draw pictures or diagrams to help me solve a math problem, I try to draw it to scale.

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<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
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<td>2</td>
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</table>
6) When I draw pictures or diagrams to help me solve a math problem, I try to label important aspects of the diagram such as angles, distances, or heights.

<table>
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<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>1</td>
<td>0</td>
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</table>

7) I find it easiest to do a word problem in math class if I translate the words into a picture or a diagram.

<table>
<thead>
<tr>
<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
<th>Strongly Disagree</th>
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<tr>
<td>2</td>
<td>1</td>
<td>0</td>
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8) When working on a word problem in math class, I find it hard to convert the words into a picture. However, once the picture is done, then the rest is easy.

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<th>Strongly Agree</th>
<th>Agree Somewhat</th>
<th>Neither Agree nor Disagree</th>
<th>Disagree Somewhat</th>
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Appendix II: Sample of Student Work
Appendix III: Transcript from Focus Group Interviews
First Focus Group Interview: Jon, Maria, and Karen

Researcher: So, um, these are kind of open-ended questions that we can discuss in a group forum, we'll see how it works. You can do your best to answer them and it's not going to be graded. I'm not going to study if you answer my questions in the “right” way or the way I'm expecting. And if you need some paper or a pencil, I brought those with me. If you want to take a break at any time, let me know, I'll turn off the camera, and you can come back when you're ready. If you have to go, then go—there's no consequences for that.

Now the first question is an open ended question. Please describe in as much detail as you can about how you go about solving math problems. Especially like, let's pretend that I just passed out a test to you guys and you just looked at the first problem. What are the steps you usually take.

Jon: I try and like, I guess, identify a formula. You know, like, you want something where you can put it into where you can start solving. So I guess a formula is the best place to start.

Maria: I usually read it over a couple times and then I think back to a problem that I've done that is also like the same as that problem. Then I kinda like parallel them to each other—that usually helps.

Karen: That's what I do too. There's some cases where like there's a lot of information, I'll try to diagram it out to see, to see if I can visualize it better.

Researcher: Okay, so, um, you guys think back to things you've done in the past, then diagram out any information that you can make into a picture. So, um, what do you see in your mind as you're first reading the problem and trying to solve the problem.

Karen: Well, it depends on the problem.

Researcher: Yeah, of course.

Jon: Yeah, that's a good one.

Maria: Well, like, really, when I first read a problem, I don't like think about, like, the correct, like, definite answer. Cuz it's more important to like know like how you got to the answer. So I always picture a lot of writing and like I'm going to write it all out and hopefully I'll get the right answer from that.

Jon: Almost picture your plan of how you're going to solve it, I guess. Like, how you're going to go about solving it. Like the steps you're gonna use.

Maria: You have to be organized too. Cuz otherwise you have to go back. You should like plan it
out in steps and lines and not like scribble everywhere.

Karen: Sometimes like with SAT problems where there's a lot of words, like you know how to do it, but you want to, like you want to find what information you need and like what information will help you get the answer.

Researcher: So how do pick which information you need? What kinds of words stand out to you?

Jon: Like in an addition problem, you know, sum or like keywords or if they give you like some numbers or something you'll be using, so

Maria: Yeah, like I just underline keywords and numbers and information

Researcher: And you said you try to organize the problem... the information?

Maria: Yeah, so like I write it down in words I can see because if I jumble it all together it's not going to work out.

Researcher: I'm just going to check and make sure this is taping because I get paranoid that I'm going to have to do this all over again [checks camera]

Jon: We can speak louder if you want.

Researcher: No, it's fine. Now this next do you ever use mental pictures to help you solve math problems? Particularly, think about when we were doing you know geometry and trigonometry.

Karen: Like sine graphs and cosine graphs?

Researcher: Yeah!

Karen: Mostly I just like draw it out before a test. Just so like just because sometimes with sine and cosine I can confuse them real easily.

Maria: Yeah, um, sine and cosine graphs are just definitely in my mind forever just because we've done them for a long time or so much. Um, as for like mental pictures—I can't just think of theorems in my head just like writing them out... it becomes much more fluent if you just write it out.

Jon: Yeah, I just picture the graph, basically. Like when we were doing like trig stuff, that's when I would just like picture it, the graph. And I guess I'm much more formula based, I like following a formula and working through it.
Researcher: Um, and what about—you said you can't really picture theorems. But what about shapes? What if you were doing a problem involving a triangle and a square.

Maria: Yeah, definitely, I think in my head. And depending on the problem, like if I need to find an angle in a triangle or something, I definitely put them in my head. But for me, it's definitely more fluent if I like write it out.

Karen: For me, I like to write it out too.

Researcher: So actually this leads to the next question. Do you work more in your mind when you're trying to solve problems, or is it more working on paper? Or is it an interchange between the two.

Jon: I'd say it's more of an interchange between the two because I guess like in my head I'm actually doing the work, but like seeing it on paper makes it clearer for me, like it helps me focus in and get my thoughts together.

Karen: More in paper, or on paper.

Maria: Same here. I always like am like doing problems regardless of if they like taught us, I always like to have my textbook so I can see it, like an example that is pretty much the same thing.

Researcher: Ok, so seeing examples of something that is similar to what you are working on?

Maria: Yeah.

Researcher: Ok. Um, okay, and then so how do you use diagrams and charts and pictures when you are trying to solve problems? Also, the other thing too, you guys don't always have to... it's okay if the interchange isn't me talking to Y and then going in a circle. If you guys want to build on what others are saying, that's okay too. So basically, how do you charts or diagrams or pictures to help you solve problems? [Pause] Or, even, do you use them?

Jon: I guess like from before, it makes my thoughts clearer and sort of get an idea of how, like I said, a visual image, to sort of conceptualize what you're looking at or what you're doing.

Maria: Um, when I use diagrams, it's usually for longer problems. For example, like in Shadows where you had to draw angles and for diagrams it just sort of clears it up a little more. So yeah...

Karen: Same thing for me. Um, oh, because somethings like you find a diagram and sometimes you've gone over it and so you like use that in an example if you sort of like forget how to do it, if you forget what you're trying to do in a test.
Researcher: So what you're saying is that diagrams are useful in multistep problems—so in a multistep problem, a diagrams is a good way to organize... So what do you guys feel like you could do better when you're solving math problems? Where are your weaknesses? What would you like to improve?

Maria: Like what part of math?

Researcher: Yeah, it could either be in the context of “What subject areas” or it could also be in the context of “When I'm in a math class, these are the kinds of mistakes I make or these are the things I could do better.” It could go in either direction.

Jon: Like for me, it's easiest for me to solve it—I guess it's easiest for anyone to solve something once they have the formula. Sometimes the more conceptual stuff sort of gets me.

Researcher: In what sense?

Jon: Uh, how do I explain this? Like, I know how to explain this in terms of science. Like, for example, in chemistry, like with the math aspect of it when we have a formula, and it's like “figure out the reaction” or something, that's easy for me. But what if you were to tell me like “Why does the chemical react?” or something, I would be like “I don't know.” You know? It was my biggest like... using the trig identities or whatever. Like I understand how to apply the formulas. Like let's say you had something in cosine, you had an answer in cosine, and you wanted to get the equivalent in sine or something like that. I could do it using the formula, but I don't know.

Karen: Also more like with proofs. So like when we're trying to prove some kind of identity, that works. I don't know, sometimes, I think sometimes identities help me to understand the concepts too. But sometimes like once you sort of go over it, you sort of understand. But then you can't really find an exact connection.

Researcher: How do the identities help you with the concepts? Can you think of an example?

Karen: I think maybe, what was it was the law of cosines and the law of sines? Was that it?

Researcher: The law of cosines the law of sines?

Karen: Yeah, those I think. I remember the proofs from that segment, and that really helped me understand how those were derived from each other.

Researcher: Okay.

Maria: Um, well, my biggest mistake during math class. I kind of feel like I have to pay attention the entire time because if I miss something I feel like if I miss something, I'm not going to be able
to catch up. So what I want to get better at is like connecting entire concepts to like other things. So...

Jon: If something is really hard to get, and like I said, going back to the conceptual part, it's something I'm having a hard time understanding, and I know like lots of other people do too, you do more like memorization than actually learning how to do something. Like you'll just memorize how somebody did the formula without understanding how or why someone did the formula. And sometimes it works and sometimes it doesn't. But I know I do that, and I know a lot of my friends do it too.

Maria: Yeah, I definitely memorize a lot, which is probably not a good way to go about solving stuff.

Researcher: Okay, that's really interesting, because. Well, never mind, I'll talk more about that later. But I feel the same way too. But, um, what about in terms of taking a test, you know, when you have a test in front of you. If something goes wrong on a problem, what is it you would like to improve on, what goes wrong?

Jon: [inaudible]

Researcher: Well, just pretend I've just handed out a test, where does the problem solving process go awry?

Maria: Well, for me, usually it's just simple algebra that can like go wrong— you know, you forgot to do this, forgot to carry the one, you didn't distribute right. That's why it's important to keep organized, because you can see right away where you went wrong. That's what usually happens for me.

Jon: Or manipulations, algebraic manipulations. Sometimes you sort of like, you don't see certain manipulations, or you don't know whether you're allowed to do some manipulations. \

Karen: Like factoring and stuff.

Jon: Yeah, [inaudible]

Karen: I guess sometimes there's just basic comprehension. Like sometimes you're not exactly sure what the question's asking. But like to ask the teacher is sort of giving you the answers. So sometimes, you know, the words trip you up.

Jon: And then there's also how sometimes teachers like to throw curveballs on tests. I mean like, they don't bring in something completely that you haven't done before, but like they really test your comprehension of the subject matter by giving you a question you haven't seen before. But it's like
dealing with similar stuff. Those, those throw me off.

Researcher: So actually, I guess I'll build on that, talking about teachers. Do you feel like there's a certain way teachers want to solve math problems, and does that differ from how you actually do it?

Karen: Well, I always want to do it... well, by understanding how this equation works, but a lot of times, we sort of just get so used to it, we just memorize. Like, we know we need to use this formula, we know that, but we don't really know why.

Maria: I've felt like in the the last two classes I've had in my junior and senior year, the teacher encouraged us to explore different ways, which is why you can have the entire class present the same problem over and over again just to see different ways to do it. So I've never felt like I've been forced to do it a certain way.

Researcher: I guess it's a hard question to answer because two of your teachers are in the room right now. But was there ever a time, even thinking back to when I was student teaching, when I was like, “We're going to solve the problem like this,” but you didn't want to solve the problem like that.

Jon: I mean, like, the only constant I can really see with a lot of teachers is that they want you to write out all your steps and sometimes you really don't feel like it's like “I don't need to show this, I understand it.” But more and more, you start to realize the significance of showing all your steps because there's less error, less chance of making an error, and stuff like that, so....

Maria: That's interesting.

Researcher: And now I'm kind of going back to visual questions—so what kind of questions are best for you to use mental pictures. You know, you were saying that you would read a problem and try to, um, turn it into a diagram in order to make it easier to solve. What kinds of questions are usually best for you guys to do that?

Karen: Well, I remember this one, it was where like, it was in chapter 7. It was with a building and it forms an angle facing the next building which was this many feet tall. You pretty much had to draw out the diagram to get the triangles that you were trying to get the angle where it faced the first one.

Researcher: Yeah, do you want to jot it out really quickly to remind everybody? I remember exactly which problem you're talking about, but if you just want to remind everybody...

Karen: [Draws diagram] You have a building facing another building, and the height is x or something, and you have this angle, and another angle here, and you have to figure out how tall this
building is....

Jon: You also have the distance between those two, right?

Karen: Yeah, yeah. Say it's 20 feet.

Researcher: Yeah, I'm glad you brought that one up. It was one of the ones I was considering bringing in. Um, are there any other... so that's a really good example of a problem where a mental picture and a diagram helps you solve it. Are there any classes of problems for example, geometry, where a mental picture helps?

Jon: Especially like trig stuff I guess.

Maria: Like the ferris wheel unit—that was very like picture and like diagram... that's what we used a lot.

Researcher: Okay, um, let's see. I was going to ask you guys if you like using pictures and diagrams to solve problems, but that seems a bit repetitive now. Um, one question: do you think of yourself as a visual learner? And if now, what kind of learner do you think of yourself as being?

Karen: I'm a pretty visual learner. It helps me to draw diagrams and graphs.

Maria: Yeah, for me, if I don't see it, I usually give up, if I'm not working with other people or like diagrams. Like for tests, if I can't visualize it, I sort of just give up on it, which is something I would like to work on.

Researcher: I guess forgot to say that there are visual learners, auditory learners, and kinesthetic learners. Auditory is when you're better with spoken instructions or a teacher giving a lecture or talking through it. Kinesthetic is when you actually do it—instead of the teacher telling you how to do it, they just give you the problem and let you go to town on it.

Jon: I don't think I'm clearcut any one of those. I don't like lectures, but I like when the teacher goes through each step and explain it. And then I also like to have a picture and a diagram. And then, um, at the same time is it kine.......

Researcher: Kinesthetic.

Jon: Actually doing it, sometimes I like to do that too, because sometimes when I actually get there, I feel pretty good about it.

Researcher: Does anybody have anything else to add?
[Students nod, indicating that they don’t.]

Researcher: Well, then this is good timing for moving on to... okay, so this is a problem from a trig unit we did. It’s from a test you guys did. If you want to take out a pencil. What I was thinking we would do is as a group, maybe one of us would read this aloud, then we could take a minute to think about it. And then we can talk about how we might attack the problem. You know, I asked you questions about “Please go into as much detail as you can about how you go about solving math problems.” That’s a good enough question, but it’s much better to just do it and talk about it as we go through.

Maria: Want me to read it?

Researcher: Sure.

Maria: Okay, suppose that you are $x$ feet away from a building looking at a rectangular billboard on the wall of a building. The bottom of the billboard is 15 feet above the ground and the billboard is 6 feet tall from bottom of top. Let theta be the angle in radians from your line of sight to the bottom of the billboard and your line of sight to the top of the billboard, assuming that your eyes are a height of 5 feet. Find a formula for theta as a function of $x$.

Researcher: Okay, maybe what we’ll do now is we’ll read it though silently again to ourselves, and then we will start talking about the problem and working through it.

[1 minute pause]

Researcher: So do you guys remember doing this problem?

Responses: Yes. Vaguely. [Head nods].

Researcher: Do you happen to remember what your plan of attack was when you did it the first time?

Jon: The first time? I remember I got it wrong.

Maria: Just drawing it out was the best way to go.

Researcher: As you were reading and before you wrote anything down—what information do you already have about this problem?

Jon: Um, numbers: 15 feet above the ground, 6 feet tall from top to bottom.

Researcher: And can you guys already picture what everything looks like?
Maria: I think so.

Researcher: Okay, so let's maybe let's try drawing it. And maybe I'll try to maybe remove myself a bit from this and let you work through it. But make sure you talk about the steps you're taking that so we can maybe capture that on the video.

[Students begin working on the problem, all three of them start out by drawing a diagram. They don't interact with each other during this preliminary work. Most students seem to draw for a while, go back and read a part of the problem, and then draw some more. All the students finish after 2 minutes]

Researcher: Okay, so if you guys want to talk about your diagrams after everybody is happy with their work. Maybe you guys can compare what you've done.

Jon: So we pretty much all have the same picture—we have theta, \( \theta \).

Researcher: So where's theta, first of all?

Jon: It's the angle between the top and the bottom of, of what you're looking....

Maria and Karen: Your line of sight.

Researcher: Okay, so you all have theta in the same spot?

[Students say yes]

Researcher: Okay, so that was transferring from what was written in the problem to the actual picture. Um, and then, let's see... so does everybody have.... how tall is the person?

Jon: Five feet.

Researcher: So where does the line of sight start? At what height?

Maria and Jon: At five feet.

Researcher: Okay. And then... so does everybody think this problem would have been a lot easier to solve if you were given the diagram right away?

Jon: Not really because you can figure out the diagram on your own pretty easily and none of us really had a problem coming up with it.
Researcher: Okay. So it says find theta as a function of $x$—that's our goal in this problem. As you guy were drawing the diagram, were you thinking about that or were you just trying to get the situation translated to a picture?

Maria: Um, I was just trying to think of the situation first and drawing it out before I thought about any calculations.

Researcher: Have you guys thought visually about what that means—theta as a function of $x$? Well, interpret that for me—what does that mean, theta as a function of $x$?

Karen: So you're trying to solve for theta with like a formula involving $x$.

Jon: So $x$ equals theta [inaudible]

Researcher: And visually, how would you interpret that?

Maria: Um depending on how far away you stand from like the billboard, is how the angle of the line of sight changes as you go closer or farther back?

Researcher: So how would it change if you were getting closer to the billboard.

Jon: It would get bigger. [N agrees. Both looking at their diagrams]

Maria: Yeah...

Researcher: And as you get further away, what happens?

Maria: The angle gets smaller. [N looking at her diagram]

Researcher: Um, should we try setting up, well what would your next step be now that you have your diagram?

Jon: I would try the laws of sines and cosines and all that stuff...which I don't remember.

Researcher: What sorts of shapes were we studying a lot during that unit?

Karen: Triangles.

Researcher: Okay, so how would we involve some triangles in this problem?

Jon: You have one right here... and....
Researcher: Okay, so Y sees two triangles. Why don't you show us and we can see if they are triangles.

Jon: Maybe I'm wrong, but this is pretty much a triangular shape. And then the distance from the ground to his body... I guess I'm wrong.

Researcher: So are you saying we have two triangles or we can make two triangles?

Jon: It's the way I drew it, I messed it up.

Karen: I think you can make a triangle. If you like took this line, I'm not sure if that helps. If this was on the test, I would try to see whatever could possibly work. I identified this triangle first with \( x \) as the base. I'm not sure if it helps.

Researcher: Okay. So this is moving back a few steps, but as you guys were drawing the diagram, how did you go about going from this paragraph here to the diagram you have now.

Jon: Well, first of all—this is just me—with this specific equation, I see that it says that the bottom billboard is 15 feet above the ground. It sort of reminded me of the ferris wheel and the legs of the ferris wheel and whatever And then the 6 feet tall part. So. I don't know, I pretty much like came back to the first sentence of the end.

Maria: For me, I pretty much went in order of what they like gave me first.

Researcher: Okay, so you saw 15 feet, so you put in 15 feet, and then you saw 6 feet for the billboard height so you put the billboard in. So step by step?

Maria: Yeah.

Researcher: Okay. Did anybody read this and then have most of the diagram already pictured and then transferred the finished picture from their mind onto the paper? Or was it better go step by step?

Maria: I just went step by step.

Jon: I had the picture.

Researcher: Okay, so you had the picture here [points at head] and then the whole picture was transferred.

Karen: This time I visualized it first, but most times I would have just done it first instead of visualizing.
Researcher: Okay. Um, do you think it might have been easier because we had some extra time to think about it before actually attacking the problem?

Karen: Um, yeah that probably made a difference.

Jon: Yeah, a lot of times on a test, the timing makes a big difference.

[Break in tape here—after break, just C and Y]

Researcher: So here's another one. We'll just do the same thing again because with the last problem, I didn't actually want to get the formula, I wanted to see how the diagram looked. So if you just want to read the problem aloud.

Jon: Alright, in order to measure the distance across the lake from A to B, the surveyor is able to measure from the third point C on land. The distance AC is 100 meters and BC is 120 meters. He finds that angle ACB is 47 degrees. Find distance AB. The figure is not drawn to scale. Okay, so I probably need to reread it once or twice.

Jon: Are we allowed to assume that this is a right angle?

Researcher: Well, I guess I'll let you decide that.

Jon: I guess not, because...

Researcher: How come?

Jon: Well usually no matter what if it is a right angle, they usually give it to you like that [draws right angle symbol on paper]. So I'm assuming that it isn't. Well, I'm assuming that I can't assume that it is, if that makes sense.

Researcher: Okay.

Jon: So...

Researcher: And you do have your book, so you're welcome to...

Jon: Can I?

Researcher: Yeah. I'm happy to have you go through it. But if we could talk about the problem solving process as you're going through...
Jon: So basically right now I'm trying to think. Alright, so I have the picture you know from the test. And I have an angle and I have two sides, so I'm thinking the Law of Cosines. I can't remember exactly what it was... maybe the Law of Sines.

Researcher: And what do you remember about the Law of Cosines?

Jon: It helps you solve for a missing side given an angle in between two sides.

Researcher: Alright, and what do we have in this situation?

Jon: Exactly that—we have an angle and two sides and we're trying to find the opposite side.

Researcher: Okay.

Jon: [Reads and copies from book.]

Researcher: Okay, you've just written down the Law of Cosines.

Jon: Well, this is the formula. And then c squared is... [inaudible]. So a squared would be... [inaudible].

Researcher: Okay, so you're replacing the a squared in your formula with 100 squared. How come?

Jon: Well, I'm trying to think of it like the Pythagorean theorem because a squared plus b squared equals c squared. And that is a plus b plus c with [inaudible]. [Assigns a, b, and c].

Researcher: Okay.

Jon: I usually have to think of c as the hypotenuse. Maybe that's an improper assumption because I made that mistake last time.

Researcher: So in this case, you're identifying c as which side?

Jon: Uh, the thing is like, I remember this question. And I remember I got it wrong, so I'm trying to think of what I did wrong—what was the error.

Researcher: Okay.

Jon: [Works on problem. He is speaking as he is reasoning, but his voice is inaudible.]

Jon: I'm sorry, I just forgot all this.
Researcher: That's okay, let's just talk about what you've done so far.

Jon: [Inaudible]

After this point in the interview, we just discussed how you could solve this problem with a calculator. We went back in the end and looked at the phrasing of the problem and noticed that we completely disconnected from the words once we had a diagram.

**Excerpt from Second Focus Group Interview:**

Researcher: Describe in as much detail as you can about how you go about solving a math problem. Like, pretend I've just handed out a test to you and you're looking at the first problem on the test.

Harold: Well, I mean sometimes it's pretty straight forward what you have to do. Just the way the problem's set up. They tell you something to do, you figure it out like... let's say they gave you a couple sides of a triangle and want you to find the Law of Cosines. Like they give you one angle and two sides and they want you to find the other side. Then it's pretty obvious what you have to do, right? I mean they give stuff like situations and stuff. Like the billboard... or the skyscrapers ones and they both make a shadow and stuff. Well, yeah, you couldn't really just look at it and be like “Oh, I know what this is and what that is.” So if it's like tough to visualize, like if it's like a situation kind of thing—they give you a situation and you don't know what's going on what they want you to find, then it's usually easier to draw a diagram and just map out where the problem's telling you where all the information is and you put all the points there. Then you just kind of sit there and stare at it and sometimes it just comes to you, you know like “You can use the Law of Sines here!” Or you could do this or that. But it's really chapter-specific too. Since it's chapter-specific, if there was a test that was like chapter 7, you pretty much narrow down all your like choices of attack or whatever. You pretty much narrow it to all the concepts that's in that chapter. So it's a little easier, you just gotta pick and choose and decide which to apply.

Letitia: I agree with that, but with me—I don't like math and I panic every time I get a math test, because I keep my grades very much in mind. So I underline what's important in a problem what they ask me to do. I draw diagrams. It's pretty much step by step what do I have to do to finish the puzzle or something.

Harold: Oh yeah, that's a good point. Like in chemistry it's all pretty much math and stuff. Sometimes I mean if you can't remember what the formula is or whatever for convergence or moles or stuff like that to a certain amount of units, sometimes it's just easier to just like write down all the information they give you and the units with them and then you can figure out how you can play with those variables, those pieces of information, and figure out how you can use them to get the units you want. That sometimes helps too, I used that on the AP test, so...
Researcher: When did you take the chemistry test?

Harold: Yesterday.

Letitia: I heard it was bad.

Harold: It wasn't too bad, but it was bad. Especially section 2, part 3.

Researcher: Do they give you written response...?

Harold: Yeah, but the problem is that they don't give you enough time. I didn't think there was enough time to answer all of them because they wanted you to explain.

Researcher: So what happens between, so pretend I have handed out a test, and you read the question, what happens in your mind after reading the question but before writing anything down.

Letitia: Panic! Panic and then calming down and seeing what the problem gives.

Harold: Maybe an example... I don't really focus on like what I do for sort of a, oh okay, so I see this is what the problem is asking and this is what I do. But if you really want to know what really goes on, maybe if you have a practice problem or something.

Researcher: Oh yeah, and we're going to do that later. But for now if we talk about generalities, and we're going to actually do it for real.

Harold: Because I don't really know what happens—you just look at the units. If it just isn't coming up, I just write down everything they give me and I see “They give me this, this, and this.” If it's a formula problem, I can see which specific bits of information I can use. And if that doesn't help, then I try reading slower. And if that still doesn't help, then I draw it out. I really don't know what happens.