How students invent representations of motion
A genetic account[☆]

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Abstract

The purpose of this paper is to illustrate, through example, a particular approach to understanding how students learn to use scientific and mathematical representations. Much of the work in these fields has been focused on a microanalysis of how students use specific representations, such as line graphs and equations, and on the particular difficulties that students have with these representations. Here, in contrast, I will illustrate an approach that attempts to place students' work with representations in the context of the broader history of their representational experience, and the capabilities that this experience engenders. I refer to this approach as "genetic," because it attempts to understand episodes of representational learning within the broader context of the genesis of representational competence. I will illustrate this new program by looking at episodes in which students are engaged in the invention of graphical representations of motion. The heart of my analysis will be the identification of what I call *constructive resources*, basic and somewhat generic resources that contribute to a student's ability to invent representations of motion. The analysis will include a number of core observations. I will argue for the importance of prior experiences with drawing and that some features of drawings are carried over to representations invented by students. I will describe a type of representation I call a *temporal sequence* that I believe develops from student experience with notational systems. Finally, I will discuss how students' abilities to make use of certain features of elements of a representational display contribute to their ability to invent novel forms. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The purpose of this paper is to illustrate, through example, a particular approach to understanding how students learn to use scientific and mathematical representations. Much of the work in these fields has been focused on a microanalysis of how students learn to use specific representations, such as line graphs and equations, and on the particular difficulties that students have with these representations. Furthermore, to the extent that educational researchers have attempted to understand these difficulties and place them within a larger theoretical analysis, the researchers involved have frequently adopted what I will refer to as a "symbol systems" perspective. In the symbol systems perspective, the use of an external representation is described in terms of the making of correspondences between a "representing" and a "represented" world. Difficulties that students have are then understood as failures to appropriately make these correspondences.

This family of approaches has been very fruitful. Especially in mathematics, a great deal has been learned about the difficulties that students have with particular representational forms. However, this focus on difficulties, and on symbol systems-based analyses, tends to obscure the fact that students enter these disciplines, at the elementary as well as the advanced level, already knowing a great deal about representations. From the time they are infants, young children are exposed to a wide variety of representations. Thus, whenever students learn to use a new representation, it is learned against this background of experience. What is missing in our accounts, I believe, is an attempt to see the learning of representations in this broader light. I refer to the new approach as "genetic," because it attempts to understand representational learning within the broader context of the genesis of representational competence.

My primary emphasis in this paper will be to illustrate this new approach, in which student learning of representations is understood within the broader context of their representational experience. For this illustration, I will focus on a particular context: students learning to construct representations of physical motions. Furthermore, I will not look at cases in which students are learning to use standard representational forms, such as line graphs, to represent motion. Instead, I will be looking at cases in which sixth-grade and high school students are asked to *invent* their own representational forms.

This shift to looking at the invention of novel representational forms is in accord with the larger shift in attitude I propose. This is true for a few reasons. First, the observation that students know a great deal about representations makes it plausible that they may have the ability to invent novel forms. Second, asking students to invent new representations can give us a wider window into their relevant representational knowledge. If we provide them with a specific representation and ask them to use it, we are greatly narrowing what we can learn about this knowledge. Finally, as I will discuss below, the shift in attitude suggests that learning-by-invention may very well be a sensible instructional approach, and is thus worthy of study in its own right.

In short, my hypothesis is that, through the wide range of experience that they have in their everyday lives and in school, children and adults acquire a collection of knowledge and abilities related to the creation and use of representations. And, furthermore, some of

this knowledge is relevant to the learning of unfamiliar representations in mathematics and science. In this paper, my goal will be to characterize part of this knowledge: I will attempt to identify some of the core capabilities that contribute to a student's ability to invent representations of motion, including some basic "ideas for representing" that can feed into the invention of novel representations. I call these contributing capabilities *constructive resources*.

Much of this paper will be organized around a discussion of constructive resources. As the paper proceeds I will, one-by-one, discuss the constructive resources that are most important for the invention of representations of motion. In addition, as I discuss each constructive resource, I will also address two important issues:

1. *Genesis*. Once constructive resources are identified, there are a number of questions we can ask. One of these questions is: What types of experience led to the development of this resource? One goal of this paper will be to attempt to answer this question; as I discuss each constructive resource, I will speculate about its developmental origins.

2. *Continuities and discontinuities*. Second, I will discuss how the existence of each constructive resource bears on the learning of standard scientific forms, such as line graphs. One of the presumptions of this work is that much of the everyday knowledge about representations that students develop is functional; it develops because it is useful in some context (Smith, diSessa, & Roschelle, 1993). However, the same resources that are productive in some contexts might pose difficulties in the learning of specific scientific representations. To frame this issue, I will speak of "continuities and discontinuities." The point is that, in some cases, students can build smoothly and seamlessly on existing representational capabilities. But, in other cases, there may be "discontinuities," and the transition may be awkward.

The notion of constructive resources, plus an emphasis on the two issues of genesis and continuity, characterizes the approach that I intend to illustrate in this paper. When I apply this program to discuss the constructive resources that are important in the present context, I will organize my discussion around the following topics:

1. *Drawings*. It is a nontrivial observation that most students have experience with representing the world on paper in *drawings*. This experience contributes a collection of resources applicable to the representation of motion. Drawings can be modified and extended by students to deal with the requirements of a scientific representation of motion, but the path from drawings to scientific representations is not always a smooth one.

2. *Temporal sequences*. A variety of student representations take the form of linear sequences of distinct elements: a list of items, written in a line, that tell the story of the motion one item at a time. I will hypothesize that the relevant resources could plausibly have been abstracted from a child's early experiences with notational systems, especially text.

3. *Features of a line segment.* Finally, I will discuss how the features of a line segment, such as its length and orientation, constitute important resources for the construction of representations of motion.

In the remainder of this introduction, I will first say a little more about prior research on the learning of representations in mathematics and science. Then, I will describe the context of this work, including the relation of this work to the larger goals of Project MaRC, as well as the specific task and settings to be focused on in this paper. Finally, in the last part of the introduction, I will attempt to give the reader a better understanding of the program I am proposing, by briefly illustrating the nature of the account I will provide.

1.1. Prior research: the symbol system perspective and learning to use representations in mathematics and science

As I stated above, my purpose here is to illustrate an approach that shifts our research in a new direction. To argue for this approach over others, and to show that it is truly novel, would require that I systematically describe what has been done. I will, however, not attempt to do that in this paper. Rather, the great majority of my effort will be directed at describing a particular enactment of my program. For this reason, I will content myself with a very rough sketch — perhaps, a caricature — of existing research on how students learn to use representations in mathematics and science. My purpose here will be to highlight features of my own program, more than to provide a complete picture of existing work.

With these caveats in mind, I will describe three interrelated features that characterize much of the existing research in this area, and then contrast my own approach.

1.1.1. A focus on specific representations rather than general capabilities

Much of the research on how students learn to use representations in mathematics in science has, appropriately enough, been focused around the learning of certain specific standard representations. For example, researchers in both mathematics and science education have extensively studied the learning and use of standard line graphs (see, for example, the extensive review by Leinhardt, Zaslavsky, & Stein, 1990). Similarly, researchers have taken great care to understand the various ways that students interpret (and misinterpret) algebraic equations (Kieran, 1992).

This focus on specific representations has important advantages. There are specific representations that are very important, both in mathematics and science, and focusing on them helps to ensure that we will be observing behavior that is of relevance to our ultimate instructional goals. But the absence of a broader view makes for limitations. First, if there are representational capabilities that are more general than individual representations (and I believe that there are), then uncovering these capabilities should be one of the goals of our research. A focus on narrow slices of the representational field will not allow us to easily uncover such capabilities. Second — and more importantly — this focus on individual representations, without the context provided by broader individual history, makes it very hard to understand *learning*. If the learning of specific representations depends on broader

capabilities that are already developed, then it will be very hard to understand how students learn to use these specific representations if we do not, first, have some knowledge of the general capabilities.

1.1.2. A focus on difficulties, rather than on capabilities

One of the outcomes of research into the use of specific representations is that it identifies the *difficulties* that students have with these representations. For example, in their review of the literature on functions and graphing, Leinhardt et al. (1990) list three main categories of "misconceptions."

1. Interval/point confusion. Students focus on a single point on the graph when it would be more appropriate to focus on a range of points.
2. Slope/height confusion. Students confuse gradients (slopes) with maximum or minimum values.
3. Iconic interpretations. Students sometimes interpret a graph of a situation as a "literal picture of the situation."

Like the focus on individual representations, this focus on difficulties has merit. Compiling a list of the specific problems that students have with important representations can be of great practical benefit. But this focus also has important limitations with respect to understanding learning. In order to understand the learning process and *why* students have these particular difficulties, we need the broader perspective provided by a genetic approach. Furthermore, if we want to have a good understanding of what the endpoint of learning should be, then we need to have good description of the nature of representational competence, not only a description of the problems that students have. It is just as important to know what knowledge is possessed by a competent grapher, as to know what the pitfalls are for novices.

1.1.3. The "symbol systems" orientation

However, it would really be a terrible caricature to state that prior work in this area only identified difficulties; there have certainly been attempts to build larger frameworks within which to understand difficulties of this sort. What I want to argue now is that many of these attempts adopt, either explicitly or implicitly, what Nemirovsky (1994) has called a "symbol systems" perspective.

To explain the symbol systems perspective, I will begin with an exposition by Stephen Palmer (1977). Palmer begins by stating that a representation is "first and foremost, something that stands for something else." He then goes on to say that this description implies the existence of two "worlds," a "represented world" and a "representing world," with correspondences between these worlds. Thus, he argues, to fully specify a representational system, one must lay out the following five aspects:

1. What the represented world is.
2. What the representing world is.

3. What aspects of the represented world are being modeled.
4. What aspects of the representing world are doing the modeling.
5. What are the correspondences between the two worlds.

This, in a nutshell, is the symbol systems view of representation. There are two separate worlds — one relating to the symbols and the other to the referents of those symbols — and there are correspondences between specific aspects of these worlds.

Views like Palmer's have appeared in multiple guises, designed for many purposes. One notable and widely cited version is presented by Nelson Goodman (1976) in his book *Languages of art*. Most important here is Goodman's discussion, in Chapter 4 of his book, of what he calls "The Theory of Notation." Here, Goodman begins by defining a *symbol scheme* as consisting of "characters, usually with modes of combining them to form others," and he goes on to explain that "characters are certain utterances or inscriptions or marks" (p. 131). Then, later, he defines a *symbol system* as consisting of a "symbol scheme correlated with a field of reference" (p. 143). This, I believe, is essentially a symbol systems view of representation.

The symbol-referent view has been used to help explain certain types of difficulties that students have in using external representations and, following these diagnoses, to aid in making prescriptions for instruction. This has especially been the case in mathematics instruction. James Kaput's (1987) *Toward a theory of symbol use in mathematics* is a notable example, both because it is relevant, and because Kaput's account is rich and extensive.

Kaput presents some definitions along the lines of the symbol-referent view. He begins by explicitly recounting Palmer's definition, just as I cited it above. Then he lays out some further definitions that are very reminiscent of Goodman. He defines a *symbol scheme* as "a concretely realizable collection of characters together with more or less explicit rules for identifying and combining them," and a *symbol system* as "a symbol scheme S together with a field of reference F, and a systematic rule of correspondence c between them..." Clearly, Kaput is operating within the territory of the symbol systems view.

Kaput goes on to specialize this view to mathematics in some interesting ways. He argues that mathematical symbol systems are a special kind of representational system in which the represented world is itself a symbol system. All of mathematics is then built up through a chain of reference, which begins from "primary referents," and then builds layers of symbol system on top of symbol system, each one becoming the represented world for the succeeding layer.

Kaput goes on to use this view to reinterpret much of what is known about mathematics learning. Here I will just focus on one important piece of his discussion. Kaput lists two types of "cognitive activity" associated with symbol systems: (1) *Reading* information from a symbol system or *encoding* information into a symbol system, and (2) the *elaboration* that occurs once information has been encoded in a symbol system. Kaput further divides the latter category into two types of activity, *syntactic elaboration*, in which the individual directly manipulates symbols (either in their head or on a sheet of paper), and *semantic elaboration*, in which the individual reasons with the referents of the symbols. As an example, a person might add 23 and 9 by mentally visualizing symbols and performing two-

column arithmetic (syntactic elaboration), or by directly invoking knowledge of whole-number arithmetic (semantic elaboration).

Now, with these last distinctions laid out, Kaput is in a position to understand — or at least provide a certain type of account of — some of the central difficulties experienced by students. One of these difficulties is the observation that students seem to manipulate symbols without understanding what they mean. For example, students may learn arithmetic algorithms, but not know why they work, or how to modify and apply them. In Kaput's terms, we can understand students as stuck within the representing world of the symbol scheme. They work with symbols, performing syntactic elaborations, but they do not *read information* out of this scheme to the reference field, or perform the corresponding semantic elaborations.

Kaput's analysis exemplifies the image of symbol use that underlies the symbol systems perspective. There are separate worlds — the represented and representing worlds — and a set of rules that describe the reasoning that can be done in either world, as well as the correspondences between the worlds. Furthermore, many of the difficulties that students encounter can be related to a failure to make appropriate correspondences.

1.1.4. My program, in contrast

This completes my very brief portrait of research on the learning of scientific and mathematical representations. I have emphasized three features: (1) a focus on specific representations, (2) a focus on difficulties, and (3) a symbol systems orientation. In reality, the literature in this area is varied, and any brief description would fail to do complete justice to this variety. Nonetheless, I believe that this portrait does capture a prevailing undercurrent and can help to motivate the need for the approach to be developed in this paper. In what follows, I will refer to the loosely related perspectives captured by this portrait as the "symbol systems" approaches [even though some are only characterized by features (1) or (2)].

The program I will illustrate does not share any of the features I have associated with the symbol systems approach. First, I focus on *generic* capabilities, rather than capabilities that relate only to specific representational forms. The constructive resources I will identify are the specific embodiment of these capabilities.

Second, I am specifically setting out to describe *capabilities* rather than difficulties or deficits. The perspective I am adopting is that constructive resources and other general capacities develop because they are, across some range of contexts, useful. This does not mean that these resources cannot lead to problematic behavior in specific cases. In fact, we will see that we can understand some difficulties as arising from the misapplication of resources that, across some range of contexts, are useful. But, my fundamental orientation is that constructive resources constitute useful resources.

Third, I will not be adopting a symbol systems perspective. A symbol systems analysis, like Kaput's described above, can be powerful, and I do not intend to argue that it is incorrect or unproductive. Symbol system-like approaches have been used, to good purpose, by a large number of researchers. [Refer, for example, to the edited book by Claude Janvier (1987) in which Kaput's paper appears.]

However, it is essential that we understand what the symbol systems perspective is, and what it is capable of. In particular, we should realize that a symbol systems analysis does not,

in its raw form, provide a theory of the knowledge or capabilities possessed by students. Instead, it describes knowledge only by the *function* that it must perform; it gives a particular take on what the knowledge must *do* — for example, reason in the representing world or translate between worlds — but it does not tell us what that knowledge *is* or how it accomplishes these functions.

For this reason, there are problems with the symbol systems view as a framework within which to understand the learning associated with external representations. For example, we might describe student errors as arising from a failure to "correctly translate" between the representing and represented worlds. But this does not give us an explanation of why students fail to translate correctly in some circumstances and not in others.

Furthermore, there are reasons to suspect that the symbol systems perspective might not even provide an appropriate *functional* framework for understanding human behavior with symbols. Returning to my discussion of Palmer, recall his statement that the representing and represented worlds are divided into "aspects," and that there are fairly strict correspondences between these aspects. This view implicitly assumes that the aspects of symbols that refer, the entities that they refer to, and the nature of the correspondences are all stable across time. But there are many reasons to doubt this assumption. Any represented or representing world can be "carved up" into aspects in more than one way, and there is no reason to believe that such a carving will remain constant over any extended episode of symbol use. More dramatically, based on observations of people working with representations, some researchers have argued that a more appropriate framework includes elements that could be said to blend aspects of the represented and representing worlds (Nemirovsky, Tierney, & Wright, 1998; Ochs, Gonzales, & Jacoby, 1996).

This observation has fairly wide-ranging implications. We are very accustomed to thinking of representations as things that, as Palmer says, "stand for something else," according to rules that we could find if we worked hard enough. But we should realize this is an assumption that might not hold, or, at the least, may hold only in limited cases.

I believe that a more neutral stance is required, at least initially, with regard to how representations function in human activity. We populate our experience with all kinds of artifacts and, for most of these, we would not presume that we could find correspondences between aspects of these artifacts and other entities. My proposal is to take a similar stance in studying the use of external representations: We should attempt to characterize symbol use as we find it in the world, and in the broader context of their history of use, without the assumption that we will find that representations "refer."

1.2. Developmental studies of representational competency

According to the program laid out in the introduction, one of my goals is to say something about the genesis of the representational competencies that I identify. Since I am not performing any longitudinal studies, my attempts in this regard will be largely speculative. However, to a limited extent, I can build on existing research to support some of these speculations. This existing research includes educational research that has looked at the school-based work of younger children with representations, and I will draw on that work

where appropriate. But it also includes some research by developmental psychologists who have looked at the capabilities of very young children. I will briefly describe some of this latter work since it may be less familiar to readers.

I begin with the work of Judy DeLoache who, in a series of fascinating studies, has charted some of the earliest steps in understanding the representational nature of artifacts. For example, in one component of this work, she has mapped out a variety of dramatic achievements that occur between the ages of 2 and 3 with regard to the understanding of pictures (DeLoache & Burns, 1994). It seems that, by the age of 2, children are quite skilled at recognizing familiar objects in a picture. However, strikingly, DeLoache found that the same children were limited in the extent to which they could use a picture to perform certain tasks; in particular, they could not use the information contained in a picture to find a hidden object. This is true up to the age of 2 1/2 years, at which point children seem to develop the ability to use pictures to perform tasks of this sort.

DeLoache's work documents the ability of children to understand pictures and drawings that have been produced by others, and clearly illustrates that there is some nuance to this development. There also is an extensive literature that documents the development of children's *drawing* expertise from many perspectives. Piaget and Inhelder (1956), for example, used drawing as a window into the mind of the child. Among other things, Piaget looked at the development of the ability to copy simple geometric shapes, such as circles and diamonds. Like DeLoache's research, this work uncovered a rich and nuanced development. For example, prior to the age of 3, Piaget found that no "purpose" was evident in children's attempts to reproduce the geometric figures. But, at around 3 1/2 years, an interesting shift occurred: open shapes began to be distinguished from close shapes in children's drawings and, in general, topological relations were maintained. (Squares and triangles were not distinguished from circles; they were all drawn as irregular closed curves.) Gradually, over the next few years, there was progressive differentiation of the shapes until, at around the age of 6, children were able to accurately copy all of the simple geometric shapes that Piaget tried. A number of other researchers have looked more broadly at how the nature of children's drawings changes with age, tracking the development from scribbles to more adult productions (see, for example, Freeman, 1980; Freeman & Cox, 1985; Goodnow, 1977; Kellogg, 1969; Lange-Kuttner & Thomas, 1995).

In addition to tracking the development of drawing, researchers have looked at the very early development of *writing-like* behavior. For example, Tolchinsky-Landsmann and Levin (1985) present evidence that children learn some of the key features of writing at an early age. These authors asked children aged 3 to 5 years to both draw and write some specific utterances. By the age of 4, the writings of children were distinct from their drawings: they were constricted in size, arranged linearly, and composed of units separated by regular blanks. This suggests that, through experience in the world, young children learn some of the key features of writing.

I have a few motives for mentioning this research on early development here. One motive is simply to familiarize the reader with the sort of work that has been done, and on which a genetically oriented program can build. However, I have also mentioned this work so that I can emphasize the dramatic nature of the development that occurs at a very early age. I do not

mean to imply that the development of representational competency is largely complete by the age of 4. However, I do believe that this research makes it clear that substantial learning, related to representations, occurs at a very early age. Even very young children are immersed in a world of symbols, and they begin developing representational competencies.

To readers familiar with science education work of the last two decades, this line of argumentation may sound very familiar. Researchers in science education have repeatedly noted that students come to the learning of science possessing a great deal of knowledge concerning the world. For example, children interact with moving objects, and what they learn is relevant to their understanding of formal physics. I am making a similar observation here with regard to scientific representations; children have a great deal of experience with representations from early in their lives, and this experience is relevant to their learning to use scientific representations.

1.3. The context for this work

The work presented in this paper is part of the larger project under discussion in this special issue, Project MaRC. As discussed in our introductory article, it is the intent of Project MaRC to study metarepresentational competence (MRC) across scientific topics, across types of representations, and across the range of uses of scientific representations. With respect to these larger goals, the concerns of this paper will be narrowed in a number of important respects. First, as suggested above, this paper will be concerned primarily with aspects of MRC that are particularly evident in the *invention* of new representational forms, rather than in the use of existing representations. Second, my focus will be narrowed to look at student representations of *motion*. This means I will be concerned with the representations that students invent to capture various aspects of the movement of physical objects in the world.

In what follows, I will be presenting observations from one of the Project MaRC activities that was enacted in several contexts, an activity we referred to as the "Motion Picture" task. The nature of this task differed somewhat across the various classroom situations I will discuss here; however, in all cases, the students were given simple verbal descriptions of motions, and asked to produce representations of the motion described. For example, many of the representations I will present in this paper were intended to depict a motion called the "desert motion:"

> A motorist is speeding across the desert, and he's very thirsty. When he sees a cactus, he stops short to get a drink from it. Then he gets back in his car and drives slowly away.

In what follows, I will illustrate my points with observations from three, somewhat different classroom situations, one with middle school students, the other two with high school students. A brief description of each follows.[1]

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[1] All school and student names are pseudonyms.

1.3.1. Benson Middle School

The Motion Picture task was first implemented with a class of eight sixth-graders at a private school. In this class, the activity was included as part of a year-long physics course, under the auspices of an earlier research endeavor. From the point of view of this earlier research, the primary intent of the course and the associated research was to explore the teaching of physics with a computer environment called "Boxer" (diSessa, Abelson, & Ploger, 1991). Thus, the introduction of the Motion Picture activity here was something of a sideline, and expectations for its success were originally very modest. In fact, the Motion Picture activity was viewed as only a short prelude to a more formal introduction of graphing. However, somewhat surprisingly, the activity was a substantial success; the students demonstrated a striking ability to invent a wide range of novel representational forms, and they were generally very engaged in the activity. For this reason, the activity was allowed to continue for five class sessions, and students returned to the discussion periodically throughout the course. The original set of five class sessions is discussed in detail in diSessa, Hammer, Sherin, and Kolpakowski (1991), as well as diSessa and Minstrell (1998).

1.3.2. City High School

In the second class situation, the Motion Picture activity was conducted with a class of eight students at a private high school. Again, this was part of an ongoing, computer-based physics course. Some aspects of this course are discussed in Hammer (1995) and Sherin, diSessa, and Hammer (1993).

1.3.3. Trenton High School

The third set of sessions drawn on here were not conducted in the context of a physics course, or within an ongoing course of any sort. Instead, volunteers were solicited in a public high school from students in ninth-grade "Biophysical Science," a required course for every student at the school. These volunteers were asked to come after school hours for a series of "miniclasses." In this paper I will be drawing examples from a series of three hour-long miniclasses in which seven students participated.

It is clear that, although there is some variety here in the populations studied, I will not be able to argue that this is a representative sample of any much larger population. Certainly, in future work, we should look to extend any claims made here. However, even with the limitations inherent in this sample, it will be possible to make substantial progress on interesting issues within this paper. First, the primary goal of this paper is to illustrate a particular analytic program — the genetic approach to understanding representational competence — and I can accomplish this task even if the specific claims that follow from the application of the program must be tentative. More importantly, I believe that the specific claims are extremely plausible; as the reader will see, my genetic account will not depend on aspects of the students' learning histories that are likely to vary greatly across portions of the student population, at least within the US.

1.4. Describing the task

Next, I will address how the Motion Picture task was described to the students at each of the three sites discussed above. Unfortunately, it is not possible to give a concise account of all the statements made by the teacher and students that served to define the task. In all cases, the teacher started with a fairly vague description and then refined gradually, based on the students' questions and initial attempts. In this section, I will say just a little more about how, in each session, the Motion Picture task was initially introduced.

Both at Benson Middle School and City High School, the description of the Motion Picture task depended heavily on the context set by other aspects of the physics course. As part of these courses, the students had been intensely engaged in programming simulations of physical motions in the Boxer programming environment. In these simulations, a Logo-like turtle is made to move around the computer screen in a manner that replicates the motion of some physical object. Fig. 1 shows the final display from a simulation of the desert motion constructed by a student at Benson.

In a number of respects, these simulations provided a starting point for the Motion Picture activity at Benson and City. In rough terms, it was hoped that students would conceive of the task as showing the motion in a new medium, essentially moving from the dynamic medium of computer simulation to the static world of paper-and-pencil. The final state of the computer simulations — what the computer screen looked like after the simulation was run — provided a basis for this move to paper. Indeed, note in Fig. 1 that a trail of dots marks out the motion of the car. These dots are farther apart where the car moved faster, and closer together where the car was moving more slowly. Thus, just by looking at the final state of the computer simulation, one can say much about the motion; in this case, the car started out moving quickly, slowed down sharply, then gradually sped up again.

At City, the teacher introduced the Motion Picture task by copying just such a series of dots onto the blackboard and presenting it as an example of a motion picture:

Ms. K.:   Your assignment is this: Using this as some kind of model [points to the dot trail on the blackboard] of what a motion picture might look like, what you want to do is you want to copy this down and make your own motion picture.

Orson: Like a flip-book?

Ms. K.:   No. What it has to be it has to be on one piece of paper. And what it has to do, as best as you can, on one piece of paper, is describe the desert motion.
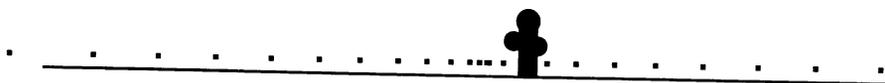
Orson: Like write it out?

Fig. 1. The desert motion simulation, as programmed in Boxer by a student at Benson.

Ms. K.:   No. It's gotta be a — look at this [points to the blackboard]. Are there any
            words here?

In contrast recall that, at Trenton, the students had not been using the Boxer environment, and the sessions were conducted after school hours. There was thus no established context on which to base the description of the Motion Picture task. Instead, the teacher was faced with doing his best to provide an abstract characterization of the task. After a brief warm-up activity in which the students were asked to invent a stop sign that did not use any words, Mr. E. gave the following introduction:

Mr. E.:   Instead of a particular sign now, the thing I want you to draw a representation
            of, like you just did for stop, is *motion*. There's lots of different motions a car
            or person could have, you could be walking slow, you could be walking fast,
            you could slow down, you could speed up, um, all sorts of different things you
            do. And, we need a way to show that, just like we need a way to show stop.
            So, when you're doing this motion thing, keep in mind we want it to be simple
            and clear, something you could explain to somebody else. Um, and something
            general enough that not only will it work for the first motion I'll give to you,
            but you could use your same idea and apply it to other motions too. And we'll
            test that and see how well it works.

The above passages only show the start of how the task was defined for students at the various sites to be discussed here. As each of the classes proceeded, the teachers gradually refined their description of the task, nudging the students when appropriate. There are many interesting questions concerning how this gradual nudging influenced what representations the students created; how the students understood the task, how the teacher guided the students at each turn, and many facets of the classroom context likely had a strong effect on the specific direction taken by students. Although these questions are an important component of Project MaRC, they will not be addressed in this paper. My focus here will be only on exploring the range of students inventive resources that are displayed, at least within contexts that are roughly similar to those in these three classroom situations. See Madanes (1997) and diSessa, Hammer, et al. (1991) for some discussion of how the teacher's instructional moves direct the Motion Picture activity.

1.5. Constructive resources

As stated above, one my goals is to account for students' abilities to invent novel representational forms in terms of what I call *constructive resources*. For an illustration of the nature of the account I will provide, consider the two representations of the desert motion shown in Figs. 2 and 3. The first one, made by Damon, is a series of pictures of a car with the speed of the car labeled at various points. It is intended to be read from left to right. The second representation, Lisa's, uses arrows of various lengths and types to indicate the speed of the car, and a large dot to indicate a stop.

How do students invent representations of this sort? Since each of these is at least partly novel, they cannot simply be following a learned set of rules or conventions in
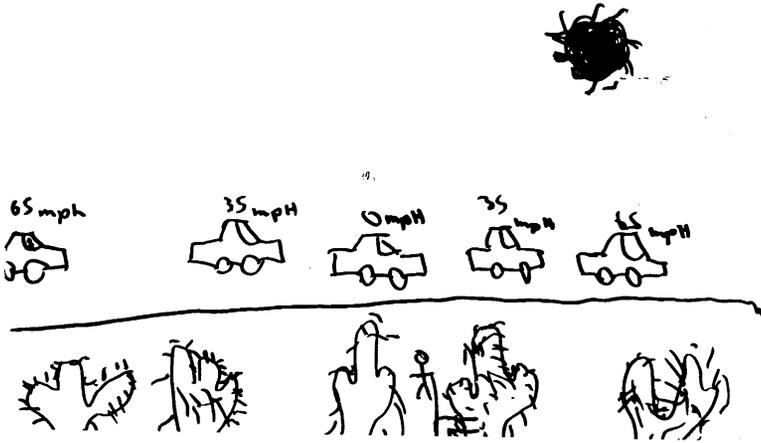
Fig. 2. Damon's representation of the desert motion. Trenton, Session 1. Scanned from original.

producing their representations. How, then, can we characterize the capabilities that underlie such constructions?

To begin to answer this question I first note that, in some respects, these constructions are not very astonishing. Clearly, neither of these representations was invented from whole cloth; there is at least some borrowing from representations that the students have used before. For example, both of these representations use *written text*. Certainly, one would not want to claim that the students invented writing for the purposes of this task; instead, text constitutes a resource that can be employed in representations of this sort. Similarly, Damon's representation employs *drawings* of a car and a sun. It is likely that he has drawn cars and suns before; he may even have standard procedures for drawing these particular objects.

The point is that, when constructing representations, students can draw on existing resources, something like a stock set of "ideas for representing." To a certain extent, we can then understand the invention of novel representations as the assembly of these stock ideas. Though this assembly is far from trivial, the availability of such resources substantially changes the nature of the task of inventing representations.
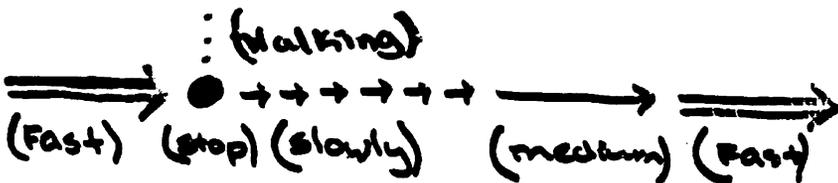

Fig. 3. Lisa's representation of the desert motion. Trenton, Session 1. Scanned from original.

I refer to this stock set of ideas as "constructive resources," and one of the goals of this paper is to begin to identify the constructive resources that are most prominent in the construction of representations of motion. As a theory of the knowledge and abilities relevant to the task discussed here, constructive resources are intended to be at a mid-level of specificity. In identifying constructive resources I intend to be indicating a collection of knowledge and abilities, but I will not attempt to be more specific about how this collection is constituted in detail (for example, in terms of knowledge structures). Furthermore, I am not at the point where I can, in any conclusive way, identify *separate and distinct* collections of constructive resources.[2] Indeed, the reader will see that there is some clear overlap in the collections of constructive resources I identify. Thus, at present, it is perhaps best to view the entire set of constructive resources as a loosely structured collection. In my discussion of constructive resources, I will be moving my lens to different parts of this collection, sometimes zooming in to focus in detail on more basic resources, and sometimes zooming out to get a bigger picture. Furthermore, these choices about where to focus my attention were not made in any particularly principled manner; rather, they were made to serve the rhetorical purposes of the paper, and to paint a picture of the larger collection.

In the sections that follow, I begin my tour of representational forms and constructive resources. I will begin with *drawing,* how it contributes to the invention of novel representations, how it can be extended, and discontinuities in the move to scientific representations. In the succeeding section, I will discuss the *temporal sequence,* a powerful resource that generates an important class of representations. Then, I will discuss how the features of a line segment — and other features of a representational display — can constitute resources for instruction. Finally, I will describe graph-like representations and the progression to graphing.

2. Drawings

Carl's representation, shown in Fig. 4, was one of the very first attempts by a student at Trenton to represent the desert motion. Many of the elements of the desert motion story are visible in this representation. The car has left skid marks, there are cacti (one with a faucet), a car, and the driver is holding up a glass of water.

> *Carl:* These lines over here [points to skid marks] I made so it'd look like um she stopped really quickly cause she saw the water. Then I had the foot marks going out to the water foun — the little faucet. And then she's in her car with her glass of water and these little things right here [points near rear wheel] are supposed to like show that she's going away pretty fast and kicking up dust.

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[2] Because each constructive resource is, in general, a collection of knowledge and abilities, I will mix plural and singular terminology; I will sometimes speak of each constructive resource (singular) as consisting of a "collection of resources"(plural). The use of the singular is to suggest some unity to the collection.
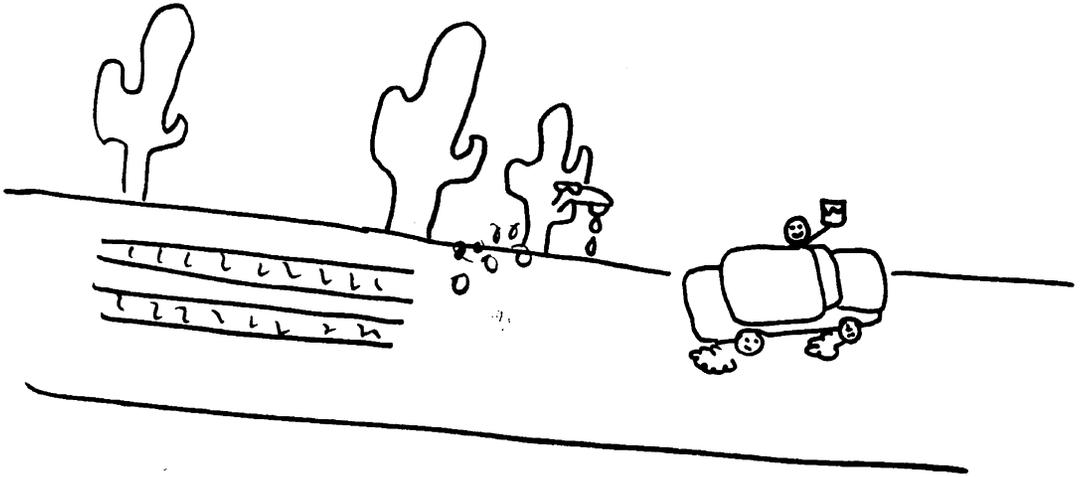
Fig. 4. Carl's representation of the desert motion. Trenton, Session 1. Scanned from original.

Looking at this figure, it seems apt to say that Carl has "just drawn a picture." Fig. 4 is certainly not one of the standard scientific representations used to represent motion, and it may be tempting to think of this variety of representation as wrong, or at least not useful for scientific purposes. But I believe that such a negative attitude toward drawing is not justified, and that there is much that the ability to draw can contribute to the use of the standard scientific representations.

2.1. Constructive resources and drawing

By "drawing" I mean to refer to a fairly wide-ranging set of general and powerful representational conventions that are learned by most children in our culture, though of course with varying degrees of expertise. "Drawing a picture" involves invoking a specific set of conventions for putting the world on paper. It includes specific procedures for drawing certain classes of objects, such as people and houses (Karmiloff-Smith, 1990; Thomas, 1995), as well as more general techniques, such as a set of conventions for projecting the three dimensional world onto the two dimensions of a sheet of paper (Willats, 1985, 1995).

The observation that students know how to draw is important and nontrivial; drawing constitutes a collection of resources that can be used in many ways in the Motion Picture task. Entire representations, such as Carl's, may be constructed so as to be more or less in agreement with the conventions of drawing (i.e., the entire representation can be treated as a picture). Drawings may also be given special modifications, such as in Damon's representation in Fig. 2, in which he has repeated the image of the car and added labels to indicate the speed. Although he has departed somewhat from the standard rules of drawing, Damon is still clearly capitalizing on extensive drawing resources. As I commented earlier, it is possible that he already knows a procedure for drawing a car,

and it has been documented that suns and sun-like shapes are early and important parts of children's repertoires (Fenson, 1985; Kellogg, 1969).

Drawing can also play a more minor role in a motion picture. The representation in Fig. 5 uses arrows of various lengths to show the magnitude of the speed, and dots to indicate a stop. In addition, it includes a small drawing of a person. This may seem like a minor addition, but note the extreme usefulness of an existing ability to draw. In this case, the students wanted to specify in their representation that the moving entity was a person. Without any drawing experience, and without an established body of conventions, it might be very difficult for students to figure out a relatively simple way to represent a person that would be recognized by others. However, these students had previous experience drawing people and, thus, it is easy for them to add a little image of a person running.

2.2. Genesis

The origins of drawing abilities are relatively clear, at least in broad outline. When they were young children, the students observed here were almost certainly given paper and drawing tools and encouraged to draw. Thus, by the time they arrived at the Motion Picture task, these students had spent a significant amount of time drawing. Because of all this time spent drawing, students have a significant amount of expertise when it comes to representing the world on paper. I cannot attempt here to explore the details of the development of drawing. As I described above, there is a significant body of research that examines children's drawing from a number of perspectives.

2.3. Continuities and discontinuities: from drawings to scientific representations

For the case of drawing, I just dealt with the first two components of my research program rather quickly; I identified one collection of constructive resources — drawing — and I spoke briefly of its genesis. It is sensible, I believe, to leave these discussions relatively brief. In this paper, I am very limited in the extent to which I can — or should — attempt to unpack the collection of resources that constitutes drawing. The educational orientation of this work determines, to some extent, where attention should be paid and the degree of theoretical specificity that is appropriate. Given this orientation, drawing must, to a great extent, be left as a "black box." Similarly, many of the details of the development of this resource are not going to be important here. We have a rough image of the kind of experiences behind this development, and that is sufficient for the analysis to be presented in this paper.

Fig. 5. Representation of a motion with two stops, one short and one long. Trenton, Round 2. Scanned from original.

Now, however, I will apply the third part of the research program to drawing: the discussion of what I am calling "continuities and discontinuities." Here, I believe a more extended discussion is merited. As educators, we need to understand the pathways that exist from drawing to the standard scientific representations: What are some typical ways that students can adapt and extend the conventions of drawing to represent motion? What capabilities that are associated with drawing can carry over to scientific representations? Where can the conventions of drawing be used productively within some scientific representations and where are these conventions problematic?

In this section, I will take steps to answer these questions. I will try to understand the manner in which drawing-related resources can be applied to represent motion and the difficulties that arise in attempting to move toward standard scientific forms.

2.3.1. Adapting and extending drawing

We have already encountered some ways in which drawings are adapted to deal with the requirements of representing motion. For example, in Damon's representation in Fig. 2, he has repeated the image of the car at various locations along its motion and labeled these images with the speed at each point. Similarly, consider Rusty's representation in Fig. 6. In this figure, Rusty is attempting to represent a slightly modified version of the desert motion in which the driver overshoots the cactus and must back up in order to get a drink. Rather than using a repeated drawing of the car on a single picture, as Damon did, Rusty has used a series of frames to represent this motion, each of which is a picture of the situation at some point in time. Using frames in this manner deals with the special problem posed by this alternative version of the desert motion, that the motion reverses over its own path and is thus difficult to show in a single image.
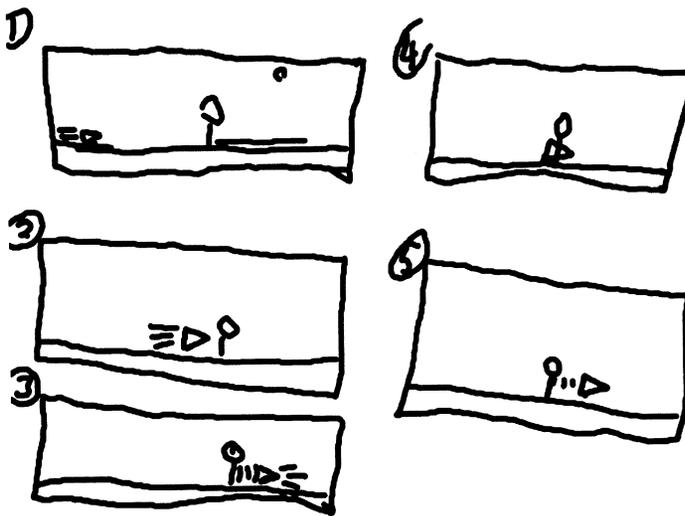

Fig. 6. Rusty's representation of the modified desert motion. City, Session 1. From video capture.

Rusty's representation also points the way to another important way in which drawings are adapted and extended for the purposes of representing motion. Note that, on four of Rusty's five frames, there are marks of various sorts that appear next to the moving object. (In this case the moving object looks like a small triangle.) When the object is moving forward, for example, there are three short lines emanating from the back of the triangle. I call these marks an "annotation." Annotations are small symbols that are added next to an element in a drawing to suggest movement.

Rusty's use of annotations is somewhat idiosyncratic; but annotations may also be used in a more or less systematic way, as in Adam's representation of the modified desert motion in Fig. 7. I will discuss this representation in more detail in a moment. For now, simply note that Adam has drawn a picture of a moving object at various points along the motion and that he has employed a number of annotations, including lines emanating from behind many of the images of the object. He explained that he used more lines and greater "blurriness" to indicate greater speed. (Contrary to the description of the desert motion given by the teacher, Adam shows the object speeding up as it approaches the stop.)

In using more lines to indicate greater speed, Adam is moving toward the use of regularized representational conventions and has perhaps taken a step toward more "scientific" representations. In fact, we can imagine a possible progression here, where other elements of the drawing are erased, and the annotations alone remain as a representation of what a scientist might take to be the essential features of the motion.

2.3.2. Limits of drawing: the "space=space" constraint

Next I will consider one of the major properties of drawing, an inherent property that greatly restricts the possible extensions of drawing to represent motion. For illustration, refer to the first frame of Rusty's representation in Fig. 6 and imagine that we were to draw something to the right of the signpost-like object. If we draw something to the right, this implies that it exists, physically, to the right of the signpost. Similarly, if we were to draw something above the signpost on the sheet of paper, this would imply that the new object was in the air above the signpost. Stated generally, a displacement of our pencil in the plane of the drawing corresponds to a physical displacement in the world of the motion.

Of course, this is only a rough heuristic and cannot be taken too strictly. Students are sensitive to particular situations in which it is sensible to violate this rule, and they have particular strategies for doing so. Rusty's frames, for example, violate this rule in a

Fig. 7. Adam's representation of the modified desert motion. City, Session 1. From video capture.

number of ways. Note that it is debatable whether it makes sense to think of the annotations as having a spatial location. Moreover, the frames themselves, because they are arranged together on a page, violate this rule; it is only obeyed within each frame. More generally, in perspective drawings, the relationship between displacements on a page and physical displacements is more complicated. Nonetheless, I believe that there is profit in thinking of some representations as, in large part, obeying the rough constraint that space = space ("space equals space"), and I will follow the implications of this constraint in this section.

To illustrate the possible importance of a space = space convention, I will discuss Adam's representation in Fig. 7 in more detail. Recall that Adam was attempting to represent the modified desert motion here, a version of the desert motion that involves a reversal in direction. The need to show this reversal posed a special challenge for Adam, and for all of the students at City High School. Just prior to Adam's presentation of his representation, Orson had presented a representation that was very difficult to read because the whole motion was shown on a single line, with the representation reversing over the first portion of the motion. Adam explained that he dealt with this issue as follows:

Adam: I wasn't able to figure out how to do — how to get Penelope [the driver] to stop and go back and then go forward again, without having what Orson had which was like — like this [draws some scribbles] and it was just [throws up arms] "what happened?" You know?

Ms. K: Okay. All right.

Adam: And so I used a ball because people always think of bouncing balls. So the ball bounced up instead of having to go back on its tracks. And uh, and then bounced away, and went on. So, that was the main thing that I used.

In this passage, Adam explains that, rather than showing a car that goes forward and then reverses, he has chosen to show a moving ball that bounces. The arcing paths near the middle of the drawing show the ball bouncing back after it has overshot the cactus, and then bouncing forward again. This is a rather striking deviation from the motion, as it is described in the desert motion story. Why has Adam decided on this modification of the story? If his description of the bouncing ball somehow serves as a justification for the representation he has drawn, why does he need this elaborate justification?

I believe that, in part, the answer to these questions has to do with the conventions that Adam has implicitly adopted. For Adam, showing the reversal posed a difficulty. He could have chosen to keep his entire representation on a single line. In that case, multiple sections of the motion would have overlapped, and the representation would likely have been difficult to read, just as in Orson's representation: "I wasn't able to figure out . . . how to get Penelope to stop and go back and then go forward again, without having what Orson had . . . " A way out of this difficulty would simply be for Adam to use more of his sheet of paper; for example, he could show spatially overlapping portions of the motion above each other, as shown schematically in Fig. 8.
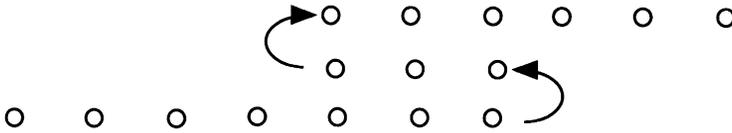
Fig. 8. Schematic for a possible way to show the reversal.

However, from Adam's point of view, there is a problem with this approach. Given the conventions that he has implicitly adopted — notably, the space = space convention — the displacements between rows in Fig. 8 must be interpreted in terms of displacements in the world of the motion. This leaves Adam with at least two possibilities. He can understand Fig. 8 as showing a top view of the motion; in that case, the figure would show the car traversing a snake-like path along the road. Alternatively, he can understand Fig. 8 as showing a side view of the motion. In this latter case, the figure shows the object going up into the air. Having chosen the latter option, Adam must rationalize the fact that he has shown the object moving through the air. This, I believe, partly explains why he invented the story about the bouncing ball.

Once again, the point here is that Adam is implicitly working under the convention that displacements on the page correspond to displacements in the world of the motion. And the striking observation is that, at least in this context, Adam would rather alter the situation to be represented than violate this convention in an unmotivated manner.

Adam was not alone in his attention to this constraint; there were many indications that this implicit convention was a strong force at City. Alan's representation, shown in Fig. 9, has a story that is similar to the one behind Adam's bouncing ball representation. There are actually two representations shown. In the top of the figure, Alan has drawn a dot trail with overlapping sections of the motion drawn above each other, much as I suggested in Fig. 8. In addition, he has also drawn an alternative representation, which shows a view of the motion looking down on the road from above. In this alternative
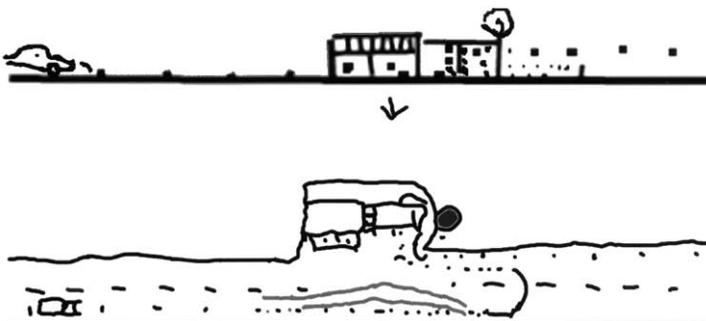

Fig. 9. Alan's representation of the Modified desert motion. City, Session 1. From video capture.

version, the car is shown turning around and moving into the other lane when it reverses direction. This, of course, agrees with the alternative interpretation of Fig. 8 I described above.

Strikingly, the other students strongly preferred the lower of Alan's representations. In the following passage, note Jeffrey's assertion that the top representation shows the car going up into the air:

Adam: The one from the side just is really — I can't get it.
Rusty: Yeah, I couldn't get that one.
Adam: I understand because you did the one from the top, but the one from the side looks like the exhaust pipe went backwards and then came back. Or something weird.
Jeffrey: It looks like he went up in the air.

Thus, we see that the students prefer a representation in which the displacement up the page has been justified in terms of a real spatial displacement in the world of the motion.

Before continuing, I want to briefly comment on the sense in which students "know" the space = space convention. I certainly do not intend to claim that a student could articulate this convention, if asked. Instead, I adopt the more limited position that there is a subset of drawing-related resources that, when deployed together, make representations that are consistent with space = space.

In summary, we see that, while drawing consists of useful resources for the construction of representations of motion, it has its limits. The restriction to space = space is a boundary that students must ultimately be able to overcome in their production of scientific representations. This is true not only for the invention of novel representations; if students were to understand instructed scientific representations such as graphing in terms of the conventions of drawing, that could lead to problematic interpretations.

In fact, there is research that could be interpreted as suggesting that this is the case. As mentioned above, Leinhardt et al. (1990) state that one of the most frequently cited errors made by students when interpreting graphs is what they call "iconic" interpretation. In iconic interpretation, a student mistakenly interprets a graph as a picture of the situation in question. For example, Leinhardt et al. cite work by Janvier (1981) in which students were given tasks that required them to interpret graphs for a car that moves around a racetrack. The most common mistake was confusing the graph with the track itself. We can understand these iconic errors as mistakenly applying the space = space convention.

This analysis is, I believe, consistent with the analysis provided by Elby (this volume). Elby attributes iconic errors to the misapplication of a "fine-grained intuitive knowledge element" that he calls "WYSIWYG" (What You See Is What You Get). It is possible that such an element can be viewed as associated with drawing resources. More importantly, Elby and I agree here on a central point: It is not productive to see iconic errors as arising from a misconception; instead, we can see them as arising from the misapplication of knowledge that, in some circumstances, is useful.

2.3.3. Getting beyond drawing

In the above discussion, I described how the students at City High School were resistant to making representations that violated the space = space convention. Next I will flesh out this story a little further, and describe how the City High class ultimately moved beyond drawing and space = space. This will complete my discussion of continuities and discontinuities for drawing.

To begin I must first introduce another important extension of drawing. Notice that in Alan's representation in Fig. 9 the path is marked out by a trail of dots, with the dots farther apart where the car is moving faster. The use of this technique here is not too surprising given the history of work at City High School. Recall that, in the computer simulations produced by students at Benson and City, the moving object leaves behind a trail of dots, with dots farther apart to indicate greater speed, just as in Alan's representation (refer to Fig. 1).

I need to take a moment to consider the properties of this ''Dots'' representation and other related representations. Note that, in Dots, displacements on the paper are still tied to displacements in the world; moving a pencil from one dot to another corresponds to moving along in space. But the space on the page is, in some respects, doing two jobs. In addition to indicating a spatial displacement, the separation between dots gives the speed; the students could look at a trail of dots and read off the speed at various locations along the motion. Thus, the Dots representation is a way of representing speed that is linked to displacement, and without violating the space = space constraint.

Other representations worked in this manner, using separation on the page to represent both displacement and speed. For example, Ryan's hash mark representation, shown in Fig. 10, clearly has strong similarities to the Dots representation. Also notable is the representation that the Benson students referred to as ''Chalk.'' (Refer to Fig. 11.) The Chalk representation uses a series of horizontal lines, with a longer line indicating greater speed. If we place dots in the gaps between lines, and then remove the lines, it is evident that the Chalk representation has similar spatial properties; horizontal displacements on the page are used to represent both physical displacement and speed.

Representations like Alan's in Fig. 9 — dot trails placed within drawings — quickly became the standard at City. We can imagine a reason for their appeal; these Dots-like representations can do much of the work of representing motion while still preserving much of drawing. To see the close relationship to drawing, consider the representations in Fig. 12, which depict a bicycle traveling over a hill. In this motion, the bicycle slows down as it approaches the top of the hill, then speeds up again as it continues down the other side. Orson's representation — the top of the two shown — uses a version of Chalk, while Kelly's uses Dots. The spatial nature of these representations is very clear here; the dot and chalk trails are stretched over the physical space of the motion.

Only a short while into the first session at City High School, the entire class seemed to settle on using a few varieties of these representations that stretched dot and chalk trails over a drawing of the motion. The teacher at City — Ms. K. — was not satisfied with having students only draw representations from this limited class. Although she judged these representations to be acceptable, she was hoping for greater variety in order to foster a more interesting discussion of the benefits and drawbacks of various representations, and ultimately
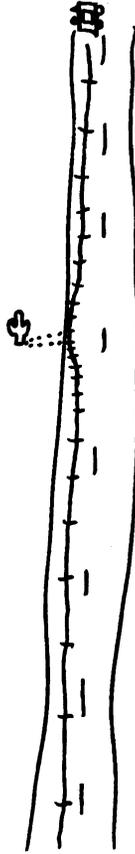
Fig. 10. Ryan's representatton of the desert motion. Trenton Session 1. Scanned from original.

to help support the introduction of standard graphing. She thus tried a number of interventions in order to encourage the City students to expand their repertoire. One particular intervention, which helped to elucidate some of the limitations of these representations, seemed to be particularly helpful.

She began by asking a student to represent the standard desert motion using the Dots representation, and then she proceeded to erase what she deemed to be "unnecessary" features, including a picture of a car, a cactus, and the road. The result was a representation of the desert motion using dots similar to the one shown in Fig. 13. Ms. K. then asked the students if this representation showed the *duration* of the stop at the cactus. The students were clear that it did not, and proceeded to argue whether it was even possible to "show time" in a motion picture. Ultimately, Alan suggested using a vertical row of dots to show the duration

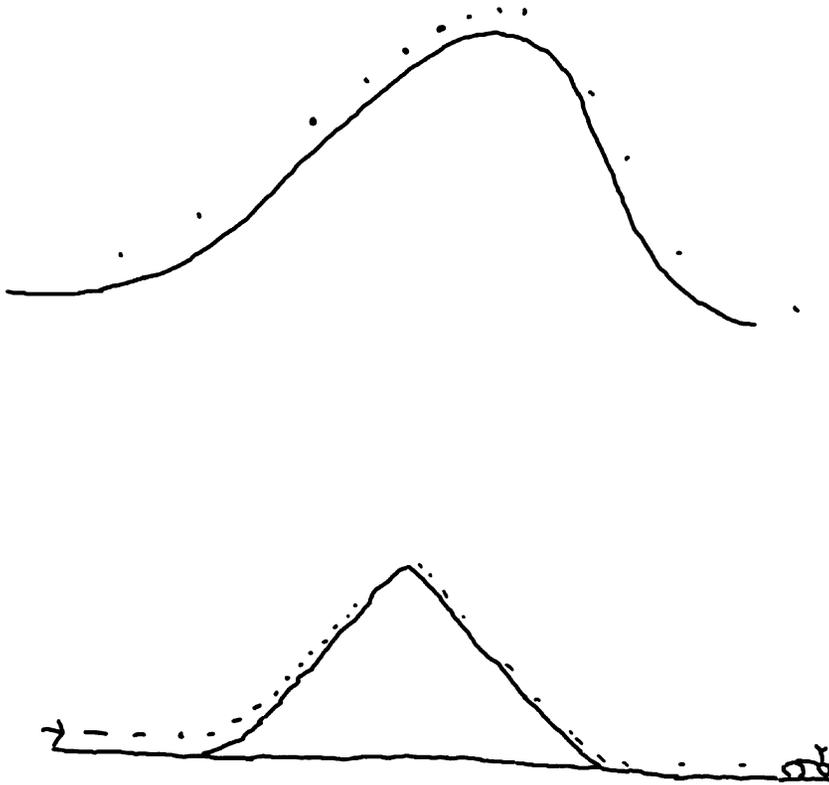Fig. 11. The Benson "Chalk" representation, our rendition.

Fig. 12. Orson and Kelly's representation of the motion of a bicycle over a hill. City, Session 1. From video capture.

of the stop, as shown in Fig. 14. He argued that, since the dots are produced at fixed time intervals, counting up the dots could give the length of a stop. Furthermore, the row of dots must be vertical, since a horizontal spread of dots would indicate continued motion, rather than a stop.

In this new representation, movement up and down the vertical line does not correspond to movement up into the air or along any direction in space. In using the vertical dimension in this manner, Alan was, for the first time at City, breaking the tie between displacements on the sheet of paper and spatial displacements in the world. Following this discussion, the range of inventiveness seemed to open up somewhat; in fact, the first graph-like representations appeared only a few minutes later.
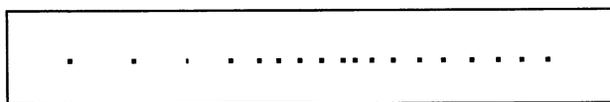

Fig. 13. A typical rendition of the desert motion using the "Dots" representation.
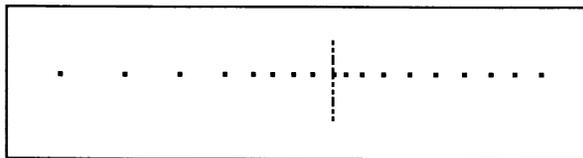
Fig. 14. Using a vertical row of dots to show the duration of the stop. City, Session 2. Our rendition.

So, we see signs here of the students at City High School breaking the bonds of drawing. They progressed first from drawing to a useful but limited extension of drawing, the Dots-like representations, and then were "stuck." Then, with some strong Socratic guidance from the teacher involving a focus on the duration of stops, they were able to progress further. This illustrates the start of a possible pathway from drawing to standard scientific representations.

3. Temporal sequences: a story in symbols

Unlike City High School, some of the early representations at Benson and Trenton departed strongly from drawings. Instead, they were based around alternative constructive resources, including what I call a *temporal sequence*. A temporal sequence is a linear sequence of distinct elements that tells what happens in a motion, one step at a time. Although this variety of representation did not show up during the early stages of the work at City, these representations do not appear to be any more difficult or more advanced; in fact, temporal sequences were among the first motion pictures drawn at Trenton. An example is Nina's first representation, which is shown in Fig. 15.

Nina's representation tells the story of the desert motion one step at a time, and is intended to be read from left to right and top to bottom in the manner of text. Nina used a pair of feet (which, here, look like musical notes) as an icon to stand for movement at any speed. Her representation shows:

1. Movement, indicated by a pair of feet.
2. A stop of 2 min, which is indicated by a labeled horizontal line. The water drops above show that she got a drink during this period.
3. Movement, indicated by a pair of feet.
4. More movement, indicated by another pair of feet on the next line.
5. Continuation of the movement, which is suggested by the string of dots.

Thus, Nina's representation is a sequence of elements, arranged linearly in the manner of text, that tells the story of the motion.

Some of the most popular and long-lived representations at Benson may be understood as temporal sequences. An example is the "Slants" representation, shown in Fig. 16, which was originally invented by Mitchell, and which was revisited by the students over the next few
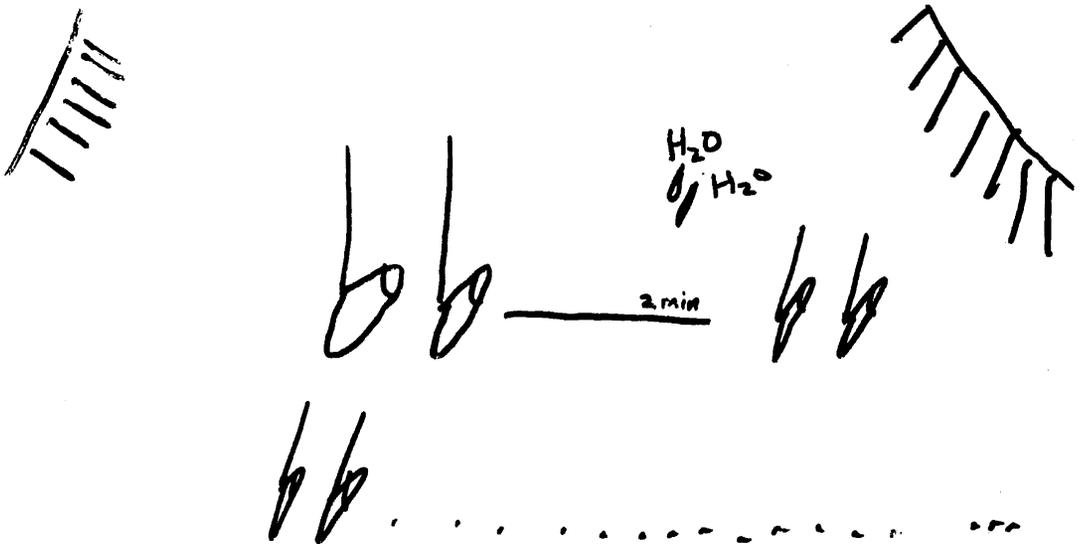
Fig. 15. Nina's representation of the desert motion. Trenton, Session 1. Scanned from original.

sessions. In this representation, the slope of the line segment is used to represent speed, with a vertical line corresponding to a stop, and a horizontal line corresponding to "as fast as the car can go." Thus, the Slants representation uses a sequence of elements to show the speed through the time of the motion.

Fig. 17 shows another example. Here the motion depicted involves two stops, one long and one short. In this representation, dots are used to represent stops, with the size of the dot indicating the length of the stop. In addition, longer and shorter vertical lines are used to represent the magnitude of the speed. This convention of using vertical lines of varying length was employed at both Trenton and Benson; at Benson, it was given the name "Sonar."

It is worth taking a moment to comment on some subtleties and argue that temporal sequences are really fundamentally different than the representations discussed earlier. Looking at the Slants representation in Fig. 16, it is not absolutely clear that the sequence of elements shown are intended to be spread out through time and not space. In fact, the first versions of the Slants representation included a line drawn below the slants that was intended to represent the road.

This ambiguity, I believe, is simply part of the nature of temporal sequences; there need not be a right answer as to whether students understand a sequence of elements on the page as distributed through space or time. We can only say that the elements tell the story of the motion, one element at a time. Thus, in calling these representations "temporal sequences" I

Fig. 16. Mitchell's "Slants" representation. Benson, Session 1 and later. Our rendition.

Fig. 17. A version of the "Sonar" representation from Trenton. Trenton, Session 2. Scanned from original.

do not necessarily mean that the horizontal dimension represents time, in some strict sense. All I mean is that the sequence of elements takes us through the time of the motion story, step by step.

3.1. Temporal sequencing, a constructive resource

I believe that it is fruitful, as a starting hypothesis, to presume that the ability to construct a temporal sequence constitutes a basic and very important constructive resource. In this case, the resource may function something like a template that guides the invention of many representations. This template is very simple; it only specifies that there is a linear sequence of distinct elements. Furthermore, once this template is invoked, a significant task remains: deciding what to fill in for each element. Nonetheless, this template provides a nontrivial basis for constructing a class of representations. Although it does not come close to specifying a representation in all of its details, it narrows the space of possibilities and provides a solid base to build upon.

3.2. Genesis of temporal sequencing

I believe that the roots of this resource are developmentally very deep. As I mentioned earlier, although temporal sequences did not appear during the early stages of work at City, they were among the first representations invented at Benson and Trenton. Furthermore, there is evidence that children develop the core abilities associated with temporal sequences at a very early age; there are instances in the research literature of young children constructing representations that are essentially temporal sequences.

An example can be found in the work of Barbara Tversky and colleagues (Tversky, Kugelmass, & Winter, 1991). Tversky asked students in kindergarten through ninth-grade to use stickers, placed on a sheet of paper, to construct representations. For example, in one task, the student was shown three stickers that stood for breakfast, lunch, and dinner. Then the experimenter placed the lunch sticker at the center of a sheet of paper and asked the child to place the breakfast and the dinner stickers. Over 70% of the kindergartners arranged the three stickers in a line to show the order of the meals, with this percentage increasing for older students. It is certainly not clear whether these kindergarten students possess all that is needed to construct temporal sequences; in fact, Tversky specifically designed her tasks so that it was not necessary for the children to invent the symbols that would go in each element of the sequence. However, these results suggest that the ability to use linear sequences of elements to represent can be traced back very early in the developmental

sequence, and thus that this ability is likely well-established in the older students that participated in our Motion Picture task.

Where is this ability to construct temporal sequences learned? My guess is that this resource has somewhat different origins than the drawing-related resources discussed earlier. Rather than originating in experiences with drawing objects and scenes, I believe that the temporal sequence is abstracted from early experiences with notational systems, such as written text. An example can be found in the work of Tolchinsky-Landsmann and Levin (1985), cited earlier. Recall that these authors asked young children to write some specific utterances and that, by the age of 4, the writings of children were constricted in size, arranged linearly, and composed of units separated by regular blanks. Similarly, Karmiloff-Smith (1992) argues for a high degree of sensitivity to aspects of the notational environment prior to any formal schooling. She notes that even toddlers make different motions when they pretend to write or draw. Since writing is an instance of a linear sequence of elements, all of this work, taken together, suggests an extensive and early basis for sequences of elements as a constructive resource.

Still more evidence for the early development of children's abilities with temporal sequences can be found in the work of Jeanne Bamberger (1991). In her book *The mind behind the musical ear*, Bamberger engaged fourth-graders in the invention of representations of simple rhythms. The result was a wide range of temporal sequence-based representations. As part of her discussion, Bamberger presents a detailed and insightful analysis of some of the more subtle issues that arise when temporally extended phenomena — in her case, music — is represented in the form of a temporal sequence.

3.3. Continuities and discontinuities

The issues of continuity and discontinuity between temporal sequence representations and standard scientific forms are somewhat different than the issues associated with drawing-based representations. Because they break the strict tie to space, temporal sequence representations can more easily be adapted to represent features of motion that are difficult to represent with drawing-based constructions. For example, we saw that representing an extended stop is problematic in drawing-driven representations that maintain space = space. This feature of motion is less problematic in temporal sequencing. Consider the representation in Fig. 18 of a motion with two stops, the second of which is twice as long as the first. Since more dots are used to indicate a longer stop in this representation, this means that each of the stops is spread out along the page. This is fine in a temporal sequence representation, but might be seen as problematic in a drawing.

In our examination of drawing, we also saw the difficulties posed for students in attempting to represent a motion involving a reversal of direction. Fig. 19 shows how this can be handled

Fig. 18. Representation of a motion with two stops, one short and one long. Trenton, Session 2. Scanned from the original.

Fig. 19. Representation of a motion that includes backing up. Trenton, Session 2. Scanned from original.

with a temporal sequence. In the motion shown, a car goes forward, stops, backs up briefly, stops for a more extended period, and then continues on. The arrows that show the backing up are intended to be read from left to right, just like the rest of the representation. This is the essence of a temporal sequence representation: A linear sequence of distinct elements, read from one end of the sequence to the other, tells the story of the motion.

There are nonetheless some features of motion that temporal sequences cannot capture, but that can be captured by standard scientific forms. For example, because of their intrinsically discrete character, it is not immediately clear how to adapt temporal sequences to show continuous changes in speed. Ultimately, we will want students to learn to use a standard scientific form, such as a line graph, that is capable of representing continuous changes of this sort. Interestingly, as we will see below, it seems that it is at least possible to make a smooth transition from temporal sequences to line graphs and, more generally, to the representation of continuously changing parameters.

4. The wonderful and amazing line segment

Stating that a representation is a temporal sequence tells us little about the nature of the elements in the sequence. The elements in a temporal sequence are sometimes small icons or pictures, each one specially designed for the role they play in the representation. This was the case in Nina's representation in Fig. 15. However, in some cases, students invent a convention that determines the nature of all the elements in a temporal sequence. For example, in the Sonar representation, a vertical line of varying length is used to represent the speed. Similarly, I presented representations in which arrows of varying length were used to represent speed (Fig. 18) and circles of varying size were used to represent the duration of a stop (Fig. 17).

To more fully understand students' ability to invent these representations, I believe that we need to focus in on a somewhat different category of constructive resource that operates in concert with temporal sequencing. The particular resources I have in mind here relate to a student's ability to associate meaning with features of a display other than the spatial arrangement of separate elements. To begin, I want to focus on how students make use of features of a *line segment*.

More than one might expect, a line segment is a flexible and powerful creature. It is possible, for example, to vary the length, the slope, the thickness, or the color of a line segment. As I just mentioned, the length of a line segment was used to represent speed in a number of the representations already presented, such as in Sonar and representations using arrows (Figs. 17 and 18). The slope of a line segment was used to represent speed in Mitchell's Slants representation (Fig. 17). In addition, a representation that I have not yet

Fig. 20. The "Eiffel" representation, which used the thickness of the line to represent speed. Benson, Session 1. Our rendition.

discussed, the "Eiffel" representation shown in Fig. 20, was described by students as using the thickness of a line to represent speed.

The students were sometimes quite articulate about their use of these features of a line segment. Strikingly, this was particularly true of the middle school students at Benson. During the third session, Mitchell proposed a modified version of his Slants representation (Fig. 16) in which the line segment did double duty: The slope of each line segment still represented the speed, but the length of the line now represented the distance traveled during the time interval in question. In the following passage, note how Mitchell explicitly discusses the independence of these two features of a line segment, and how he seems to be quite consciously making use of these features as resources for his representation.

> *Mitchell:* The slant — the thing that represents it has absolutely nothing to do with the length of the line, it's just the amount of the slant. So we can have slant — for like — slant is like the speed and the length of the line could be the distance... So we got, and the line represents two things. The slant of the line represents the speed and then the length of it represents the distance.

Fig. 21 shows another representation in which students made use of multiple features of a line segment. This representation was constructed by a group of students at Trenton as a modification to the Sonar representation. In this modified version, the slope has been used to represent whether the car is moving forward or backward. The exact amount of the slope is not relevant in this representation, only whether the slope is positive or negative.

4.1. Constructive resources: the features of a line segment

In a sense, the multiple features of a line segment constitute resources for construction; they are one of the building blocks that can be assembled to build new representations. The associated constructive resources (whatever their precise nature) allow students to make use of these features, to isolate and associate meaning with particular features of a line segment. Similarly, constructive resources can allow students to associate meaning with other types of features in a display. Speaking crudely, it is possible to use more in

Fig. 21. Representation of a motion in which a car goes backward and then forward. Trenton, Session 2. Scanned from original.

some graphical dimension to represent more of a quantity. For example, in Fig. 17, the size of a filled circle was used to represent the duration of a stop. In addition, the repetition of a graphical item can be used to represent a quantity, with the number of repetitions indicating the amount. In Fig. 18, for example, the number of circles is used to represent the duration of a stop, rather than the size of a circle. Adam's bouncing ball representation in Fig. 7 also makes use of this strategy, using more lines emanating from the ball to indicate greater speed.

4.2. Genesis

The constructive resources under discussion in this section are, in some respects, different than drawing and temporal sequencing. They are at a smaller grain-size, and likely have their origins in a range of experiences that cut across activity with a great variety of representations. Line segments, for example, are used in many different ways in many different representational forms. They are used to represent amount on a bar graph, and the slant of a line, in a sense, is employed to represent speed on a speedometer.

Because their range of relevance cuts across so many representational forms, these basic resources are truly worthy of the name "metarepresentational competence." An understanding of the various ways of employing a line segment described here is a basic building block that is relevant to creating and understanding a great range of representational forms.

Looking at the developmental literature can help us to get a sense for what must be learned and when it is learned. I mentioned above that a repeated graphical item can by used to represent quantity. A little thought makes it clear that this strategy is very common. For example, when we use tally marks, we make use of this strategy. Furthermore, there is evidence that children learn to use this strategy quite early in life. Martin Hughes studied children's invented representations of quantity by showing the children some bricks and asking them: "Can you put something on the paper to show how many bricks are on the table?" He found that, among preschool children aged 3–4 years, the most common representations were what he called "iconic" representations (Hughes, 1986).[3] In these iconic representations, the children represented the quantity of bricks by drawing an equal number of some graphical item that did not resemble the bricks. In most cases, these were simple tally marks; however, in some cases, the children drew other shapes, such as houses.

4.3. Continuities and discontinuities

In comparison to the other constructive resources, it makes even less sense to think of these basic resources as always productive or nonproductive for the learning of standard scientific forms. Certainly, the existence of such resources is absolutely essential for learning to use scientific representations. But instruction must tune where and how these resources are applied: students need to associate meaning with the right features, and at the right times.

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[3] This use of the term "iconic" may not be typical of uses elsewhere in the literature.

Take, as one example, the relationship between the use of features of a line segment and learning to read line graphs. Recall that, as described above, one of the most commonly documented difficulties that students have in understanding graphs is what has been called "slope/height confusion" (Leinhardt et al., 1990). The way that this difficulty is usually manifested is that students read off the height of a graph (its distance from the horizontal axis) when they should be paying attention to the slope. This suggests that there are essential issues around associating meaning with such features as slope; it is important that students learn to associate the right type of meaning with slope, and at the right times.

Of course, there is a big step from line segments to continuous graphs, and it is not obvious that learning to associate meaning with the slope of a segment will have any effect on the saliency of the slope of a curving line. However, it is suggestive that, at Benson, the first graphs were assembled by building directly on the Slants representation, hooking the slanted segments end to end. (This progression will be discussed in the next section and is also described in diSessa, Hammer, et al., 1991.) It is plausible that such a progression could help students to attend appropriately to the slope of the line in a graph.

5. Graph-like representations

Blatantly missing in all of the above sections are representations that closely resemble standard graphing. Students did, in fact, make such representations. Near the end of the first session at Trenton, for example, Carl and Ryan made the representations shown in Fig. 22. It is evident that there is more than an incidental resemblance between these representations and standard graphing. Not only do they use the height of a curved line to represent speed, they also have axes that are labeled in something approaching the usual manner. In fact, the vast majority of graph-like representations closely resembled standard graphing; they used axes and labels, much like the ones in Fig. 22.

Thus, we are dealing with a type of resource that is somewhat different than those discussed above; here we are seeing Carl and Ryan draw on their ability to make a specific representational form, rather than on more general representational resources. Even if the students in these various situations have never encountered representations of motion before, they have certainly had some experiencing with graphing and other standard scientific forms. This experience is relevant to their construction of representations of motion.

The story here might seem like a simple one, at least in comparison to the analyses presented above. We might be tempted to say that the students "knew how to make line graphs," and it was just a matter of their realizing that they could employ them. Certainly, it is not sensible to think of the representations in Fig. 22 as, in any deep sense, "invented." The students are probably not assembling basic resources to build something new; instead, they appear to be deploying a fairly well developed representational technique that they have previously learned. Thus, one might think that graphing must be treated as a separate element of this story, and that the production of graphs had little connection to the work on other representations that came before and after. However, I do not believe that this is the most useful stance we can adopt. Rather, I believe that there is a sense in which the progression to
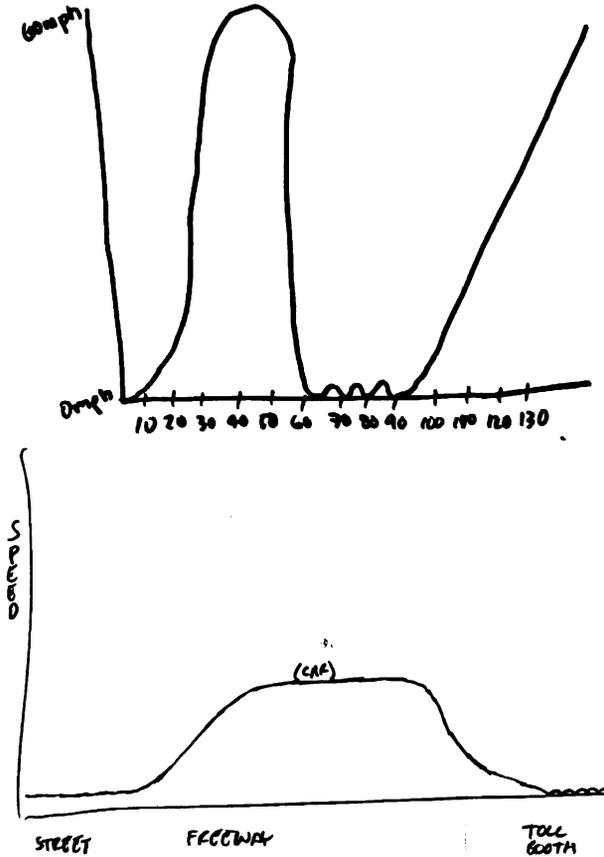
Fig. 22. Carl and Ryan's graphing representations. Trenton, Session 1. Scanned from the originals.

graphing was a bit-by-bit accomplishment, partly aided by prior representations. Furthermore, continued work with other representations, even after the production of the first graphs, contributed to students' understanding of standard graphs.

Thus, looking at how the students progressed to graphing can be very illuminating. It can help us to understand the relationships among the other constructive resources. Furthermore, looking at this progression will allow us to see some of the continuities and discontinuities playing out in the classroom; we can see the pathways from the earlier representational forms to more graphing-like forms. For these reasons, I will look at the steps that led to standard graphs at the three sites discussed in this paper.

5.1. The precursors to graphing

At all three sites there were some representations that almost certainly served as precursors to graphing, and which seem to share some of the important features of standard graphing. For example, there were precursors to the representations shown in Fig. 22, the first true

Fig. 23. Carl and Georgie's representations of the desert motion. Trenton, Session 1. Scanned from the originals.

graphs drawn at Trenton. Throughout the first session at Trenton, work proceeded in rounds. During each round, every student would first, working individually and silently, make a representation. Then, one by one, students presented their representations to the class. The representations in Fig. 22 appeared in the last such round of Session 1. Two plausible precursors, shown in Fig. 23, appeared in the just preceding round. Carl created the representation on the left by first drawing a Sonar representation (Fig. 17) and then coloring over the bars he had drawn. The thickness of the resulting wedges represents the speed. Similarly, in the right hand representation, Georgie used wedge shapes to represent the decreasing and increasing speed that is characteristic of the desert motion.

> *Georgie:*   This one is like, the wider part [uses spread thumb and forefinger of each hand to show wide spread on each side] is faster, and then it slows down like this [moves hands toward center].

Note the simple relationship between the top of either wedge shape and the line in a speed versus time graph: In both cases, a vertical displacement measuring speed is paired with a horizontal displacement that measures the sequence of a motion.

At City, there were some similar precursors to standard graphing. At one point during Session 2, Orson announced that he had a new idea based on something he had learned from music:

> *Orson:*   How 'bout like . . . it's going on the same principle like for music? Like if you're playing on the piano you have a thing which is like just to show how — What does that mean again? . . .It shows that you like go gradually?

Although Orson had trouble stating his idea, Kelly seemed to understand what he had in mind. She came to the blackboard and drew what looked like large "greater than" and "less than" symbols, as shown in Fig. 24. (Notations of this sort are used in music to indicate a place where the volume should increase or decrease.)

Following this assistance from Kelly, Orson drew the representations of the desert motion shown in Fig. 25. The first version makes sense as a representation of the desert

Fig. 24. The music-related symbols drawn by Kelly. City, Sessions 2. Our rendition.
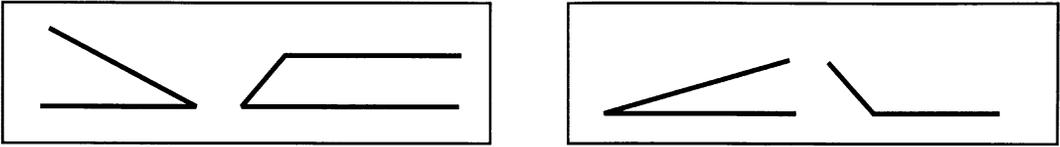
Fig. 25. Orson's graph-like representations of the desert motion. City, Session 2. Our rendition.

motion if we presume that the height of the top line above the bottom line represents the speed; however, Orson did not specify his convention. A few minutes later, Orson redrew his representation as shown in the right hand drawing in Fig. 25. Here, he essentially specified that the height of the top line above the base represented the amount of time it takes to travel some fixed distance:

> *Orson:*   So, then, right here you're going really fast, so it's less time. [indicates left edge of representation] And, as you brake, it takes more time cause you're slowing down. And here you have a stop here. [indicates gap in the center] Then here you start slowing down and then you're continuing [traces along second part of representation]... When it's up it's slow. When it goes down it's getting faster.

Note Orson was clearly inspired by an idea that came from somewhere other than standard graphing and he used it in a highly nonstandard manner (graphing inverse speed rather than speed). This suggests he has independent resources that reach important basic ideas of graphing.

Following Orson's presentation of this idea, several of the representations produced by other students closely resembled standard graphs, often employing labeled axes. The first of these standard graphs were drawn immediately following Orson's presentation, suggesting they were inspired by his representation.

Finally, I want to say a little about the precursors to standard graphing at Benson. (This is recounted in some detail in diSessa, Hammer, et al., 1991.) As I mentioned earlier, the steps to graphing at Benson began with the Slants representation. After suggesting a modified version of Slants in which the length of the line segment represented distance, Mitchell said that these modified slants could be hooked end to end, as shown in the first representation in Fig. 26. Since this drawing was only intended to illustrate a general technique, not to represent a particular motion, the teacher asked Mitchell to use his new technique to represent the desert motion. In response, Mitchell drew the second representation in Fig. 26, which employs a continuous variation in the slope of the line. It should be kept in mind that Mitchell is still using the slope of the line to represent speed, with his convention that a vertical line means that the object is stopped. Thus, this representation shows the car initially moving very

Fig. 26. The precursors to graphing at Benson drawn by Mitchell. Benson, Session 3. Our renditions.

quickly, slowing to a stop, then speeding up again. The small gap in the middle is intended to represent the stop.

Mitchell's new representation quickly inspired the other students to produce representations that closely resembled standard graphs. In fact, after Mitchell drew the continuous representation in Fig. 26, another student immediately suggested adding a grid, which was drawn over the top of his figure.

5.2. Continuities and discontinuities: why the precursors led to graphing

Reflecting on these progressions to graphing, described above, can help us to better understand the nature of the continuities and discontinuities between the early representations and graphing. To begin, it is worth emphasizing the point that, at each of the sites considered here, the progression to graphing involved multiple steps. This suggests that graphing was at least something of a gradual accomplishment in these contexts, and hints at the existence of continuities from the early representations to graphing; there are pathways from the earlier representations, through the precursors, to standard line graphs.

A closer consideration of how the precursors function can help us to better understand some of the continuities and discontinuities. First, there is still a possibility that the connection between the precursors and graphing was a weak one; in particular, it is possible that the leap to graphing was made only because of a superficial resemblance of the precursors to graphing.

But I believe that the precursor representations also aided the transition to graphing because they helped to bridge some important discontinuities between graphing and the preceding representations. Perhaps the most central discontinuity, discussed above, relates to the continuous nature of graphs. All of the other representations described in this paper were piece-wise constant; they showed finite duration segments of motion over which the speed (or some other property of the motion) was constant. In contrast, line graphs allow the direct representation motions with continuously changing speed.

In addition, the precursors and line graphs both employ lines in a way that is very different than the preceding motion pictures. Note that, as shown in Fig. 27, one can draw an envelope over a Sonar representation to generate something like a speed versus time graph. In fact, Carl's graph-like representation in Fig. 23 was generated in a similar manner, by coloring over a Sonar representation. And a trained eye could "see" such an envelope, whether it was drawn or not.

It is revealing to compare these two ways of using a line to represent speed. Rather than using lines to directly represent speed, an envelope representation uses the height of the line,

Fig. 27. Sonar representation to which I have added an envelope.

the gap between a line and the base. This alternative way of using a line may be one of the key — and difficult — innovations of the graph-like representations.

Finally, I should also emphasize that the appearance of graphs did not, in general, constitute the end of the Motion Picture task. Once the first graphs were drawn, the students did not adopt the attitude that graphs were obviously the best motion picture. At Trenton, for example, the students were strongly disinclined to draw graphs, even after graphing was clearly acknowledged as a possibility. In fact, no students would have drawn graph-like representations in the second and third sessions if the teacher had not firmly asked the students to produce a few such representations. (Given a choice, they elected to make other forms.) Furthermore, when graphing was used at the three sites, it was not used without difficulty. Thus, even after graphing first appears in the Motion Picture task, students must still do a great deal of work in order to see where graphs are useful, why they are useful, and how to make and interpret them in a consistent manner.

6. Summary and conclusion

My purpose in this paper has been to illustrate a particular approach to studying the learning of scientific and mathematical representations. I have not undertaken an analysis of how students use a specific representation and the difficulties that they have with that representation. Nor have I undertaken a "symbol systems" analysis in the narrow sense — I have not attempted to understand the difficulties that students have in terms of a failure to make correspondences between a representing and represented world. Instead, my goal has been to see students' work with representations in the context of the broader history of their representational experience, and the capabilities that this experience engenders. I looked for generic capabilities, relevant to a range of representational forms, and I tried to understand how these capabilities could support the learning of new representations, or lead to difficulties. In this regard, I hope that I have begun to show that students have a great deal of experience that is relevant to their learning of scientific representations, and that an account of existing resources is helpful for understanding what students do.

I illustrated this approach while looking at episodes in which students are inventing representations of motion. Along the way, I discussed three classes of constructive resources that contribute to a student's ability to invent novel representations:

Drawing: Drawing constitutes an important collection of resources that contributes in a variety of ways to a student's ability to construct representations of motion. I believe that this observation is nontrivial; because students know how to draw, they know a great deal about how to represent the world on paper. Furthermore, I believe that the origins of drawing in student experience is relatively clear: Most people in our culture have at least some experience with drawing as children.

I also discussed the pathways and connections between drawing and scientific representations. Some student representations are constructed so as to be more or less in agreement with the conventions of drawing. In addition, drawing can be extended and modified to represent motion. I also discussed an apparent limit of drawing, described by the space = space

constraint, which can pose difficulties in the move to scientific representations. If students get "stuck" on making representations that are consistent with space = space, this limits the range of forms in crucial respects. Furthermore, if standard forms are understood in terms of space = space, this can lead to inappropriate interpretations, such as the "iconic" interpretations of graphs mentioned above (Leinhardt et al., 1990).

Temporal sequences: Some representations took the form of temporal sequences, linear sequences of distinct elements that represent a sequence of happenings. I hypothesized that this resource could plausibly have been abstracted from a child's early experiences with notational systems, especially text. In addition, I discussed evidence that, before children learn to read and write, they are sensitive to just these features of writing.

Features of a line segment: I discussed the many ways that features of a line segment, such as its length, orientation, and thickness, constitute resources for construction. With regard to genesis, my hypothesis was that the ability to make use of these features develops from a wide range of experience with representational displays.

Graph-like representation: Finally, I noted that students sometimes just made standard graphs, apparently employing a previously learned technique. I also discussed the pathways to graphing, which passed through what I called "precursor" representations.

6.1. How sure can we be about specific constructive resources?

The categories and resources described in this paper constitute working hypotheses. Even if the constructs presented are internally consistent, and even if they provide an interesting lens through which to view the data, this may not be good enough. My hope is to eventually settle on constructs that I believe are psychological or socially real; I want to "carve nature at its joints" when describing student abilities and progressions in the design activity.

This will ultimately require triangulation across multiple kinds and sources of data, including the miniclasses presented here, classroom studies, and laboratory work. The observations at City High School can provide a hint of how such triangulation might proceed. At City, we saw that the students appeared to be "stuck" on versions of the Dots representation; for a while, the students at City produced these representations to the exclusion of others. If we find, across classrooms, that groups of students tend to work within a class of representations or to employ only a certain limited set of resources, this adds to the likelihood that we have identified psychologically or socially real constructs.

In addition, developmental evidence and argumentation can help greatly in triangulating on constructive resources (diSessa, 1994). If we can identify possible origins for resources — a hypothetical developmental trajectory — and we can find evidence for this trajectory, then this adds greatly to the plausibility of our hypothesized resources. In this paper I have tried to support my hypotheses with such arguments.

6.2. Instructional implications

As suggested above, a central purpose of this paper has been to contribute to theoretical perspectives on the learning of representations. But I have not been guided solely by a

concern for basic research for its own sake. I have attempted to target the theoretical analysis in this paper at a level that is educationally useful, and I believe that this work can also feed into the development of interventions that improve instruction in the use of scientific representations. One way that this work can contribute is that the analyses presented in this paper can help us to understand some of the difficulties that students have with standard representations (such as the difficulties described by misconceptions research) and thus could potentially help us to adjust existing instruction. For example, I discussed above how this work can contribute to our understanding of two of the most frequently cited problems that students have with graphs: "slope-height confusion" and incorrect "iconic" interpretations.

More importantly, I believe that the sort of analysis presented here is critical to the design of innovative instruction based around student-invented representations. One of the claims of Project MaRC is that engaging students in a process of invention can be a powerful way to teach them about scientific representations. This claim rests on the assumption that it would be beneficial for students to know more than the set of rules and conventions that define a particular scientific representation. In addition to these, we would like them to know why each particular representation takes the form that it does, what each representation is suited for, and why it is better (or worse) than alternatives for particular tasks. Even if students are strongly guided in their attempts to invent representations, and even if they already know a great deal about standard forms such as graphing, we believe that it is still possible for them to address these important issues as they strive to invent new representational forms. Furthermore, the recognition that students come to science instruction already possessing a great deal of knowledge pertaining to representations helps to suggest that tasks like the Motion Picture activity may be plausible instructional alternatives.

We expect other benefits from instruction that gives students the opportunity to invent their own scientific representations. We believe that, by giving students the opportunity to express their creativity, we can provide an entry into scientific endeavor for students who might otherwise not be attracted to science. In addition, inventing representations can do more than help students to understand existing representations; the ability to invent representations can be a useful skill in its own right. This is particularly true given the increased prevalence of technological tools. The wide availability of computers has allowed an expansion in the quantity and variety of representations that people are faced with on a daily basis, and in the opportunities for individuals to create their own representations. We want students to have the skills to negotiate this representational fray, and themselves to become disciplined designers of representations.

However, as discussed in Madanes (1997), activities like the Motion Picture task place strong demands on a teacher. The teacher must be prepared to respond flexibly to a wide range of student inventions and ideas. It requires, at a minimum, that the teacher know some of what to expect as the class proceeds: what kinds of inventions students might offer, which inventions offer productive avenues to follow, and how to help guide student work in these productive directions.

Research of the sort presented here can help. If we can provide teachers with a reasonably good idea of what to expect, their burden will be reduced. The "map" of student inventions

and the possible pathways provided here constitutes a start on the foundational material that is required.

Finally, I should be clear that this focus on invention does not mean that we believe students should not be taught the standard scientific representations. It is absolutely essential that students develop a familiarity with the conventional forms. Nor do we believe that students must reinvent the standard representations on their own. Rather the point is, if the invention of representations is part of what students do, then they may develop a fuller understanding of the purposes and underlying rationale of the conventional scientific representations.

6.3. What's missing in this analysis?

The particular analysis presented in this paper leaves out some important work. Most prominently, I have not tried to present a thorough account of the *process* of invention: why students make the representations that they do, when they do. Whenever a student invents a new representation, what they make depends on many things other than the constructive resources they have available, such as how the task was specified, how the student understands the task, what representations have already been made, teacher interventions, and social constraints. (Refer to Granados (this volume) for analyses that are more in this line.)

Ultimately, an analysis of the design process will be essential for both theoretical work and the design of interventions. For the theoretical work, we would like to know how what we are seeing of student abilities is particular to the limited contexts studied and the particular nature of the Motion Picture task. For the development of interventions, we will need some understanding of how to guide the design process; a simple "map" is not enough. Though we did see some interesting and suggestive pathways through the design space, and I did discuss some powerful teacher interventions, much of what the various teachers did to direct the design process was left implicit. Nonetheless, I believe that the work presented in this paper constitutes a central component of the larger program.

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